The Number of Saturated Actuators and Constraint Forces During Time-Optimal Movement of a General Robotic System

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***Abstract*—In this paper we prove that if the dynamics of a general robot system are defined by n coordinates, m differential constraint equations, and p actuators, then some combination of at least $L = m + p + 1 - n$ of the actuators and internal constraint forces are saturated during a time-optimal movement of the system along a prescribed path. The result applies to a general class of dynamic systems with both holonomic and nonholonomic constraints.**

I. Introduction

The movement of a robotic system from one position to another along a prescribed path can be accomplished in minimum time by maximizing the acceleration or deceleration along the path, subject to limits on the torque that can be applied to the actuators; see [1], [10]. In this paper we show that, for general robotic systems, the maximum or minimum acceleration is determined from a linear programming problem. The fact that the solution of a linear programming problem is an extreme point of the set of feasible solutions allows us to compute the minimum number of actuator torques and internal constraint forces that must equal their bounds during the time-optimal movement.

We believe this is a new result that generalizes the results of [1] and [10] to include a general class of constrained robotic systems such as those studied by Huang and McClamrock [5] and Mills and Goldenberg [9]. This work focusses primarily on the theoretical aspects of the problem, particularly the number of saturated constraints in a general robotic system. An example of the application of these results to time optimal control of cooperating robots can be found in [2].

II. Equations of Motion

We consider robotic systems that are described by equations of motion of the form

$$M(q)\ddot{q} + H(q, \dot{q}) = B(q)\tau + \Phi(q)^T\lambda \qquad (1)$$

where $q \in \boldsymbol{R}^n$ are the configuration coordinates of the system, $M(q)$ is an $n \times n$ matrix defining its inertia, and $H(q, \dot{q})$ is the vector of Coriolis, centrifugal, and gravitational terms. The vector $\tau \in \boldsymbol{R}^p$ defines the actuator torques applied to the system, and the $m \times p$ matrix $B(q)$ defines how these torques act on the configuration coordinates q. The matrix $\Phi(q)$ is an $m \times n$ matrix that defines the differential (holonomic and nonholonomic) constraints on the system, defined such that

$$\Phi(q)\dot{q} = 0. \qquad (2)$$

Using a standard Lagrangian formulation of the dynamics, we find that the vector $\lambda \in \boldsymbol{R}^m$ defines the internal forces in the system required to satisfy these constraints; see [4]. In addition, physical considerations yield constraints on the individual actuator torques and internal forces such that

$$\tau^- \leq \tau \leq \tau^+ \quad \text{and} \quad \lambda^- \leq \lambda \leq \lambda^+ \qquad (3)$$

where the vectors and in $\boldsymbol{R}^p$ and $\boldsymbol{R}^m$ are known bounds. Note that λ^+ and λ^- may be arbitrarily large to allow for the case of large forces generated by robots contacting the environment.

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III. Time-Optimal Control

Let C denote the set of values $q \in \boldsymbol{R}^n$ defining all configurations available to the robotic system, termed its *configuration space*. A curve $\gamma(s)$ in C defines a sequence of positions followed by the system.

Definition 1: An *admissible path* is a parameterized curve, $\gamma(s) : [0, 1] \to C$, with $\gamma(0) = q_0$ and $\gamma(1) = q_1$ as the start and goal configurations, defined such that

1) $|\gamma'(s)| \neq 0$ almost everywhere in the interval $s \in [0, 1]$, the prime, $'$, denotes differentiation with respect to s);
2) it satisfies the constraints $\Phi(q)\gamma' = 0$; and
3) it has continuous second derivatives.

The Time-Optimal Control Problem: The goal in the time-optimal control problem is to determine the movement $s(t)$, where $t \in [0, t_f]$, $s(0) = 0$ and $s(t_f) = 1$, along a specified path $\gamma(s)$, that minimizes the traversal time t_f.

Lemma 1: Given a robotic system defined by (1) and an admissible path $\gamma(s)$ in its configuration space C, with initial states $\gamma(0) = q_0, \dot{\gamma}(0) = \dot{q}_0$ and final states $\gamma(1) = q_1, \dot{\gamma}(1) = \dot{q}_1$. Then the time-optimal control problem becomes: for the system $\ddot{s} = u(t)$, find the control $u(t)$ that minimizes the traversal time t_p with $f(s, \dot{s}) \leq u \leq g(s, \dot{s})$ where $f(s, \dot{s})$ and $g(s, \dot{s})$ are the minimum and maximum values of $\ddot{s}$ subject to constraints defined by (1) and (3).

Proof: Given the admissible path, $\gamma(s)$, we can compute the position and velocity of the system, $\dot{q}$ and $\ddot{q}$, in terms of $s(t)$ as

$$\begin{aligned}\dot{q} &= \gamma'(s)\dot{s}\\ \ddot{q} &= \gamma''(s)\dot{s}^2 + \gamma'(s)\ddot{s}.\end{aligned} \qquad (4)$$

The second derivative $\ddot{s}$ of the movement $s(t)$ is constrained by the dynamics of the robotic system. Substituting $q = \gamma(s)$ and (4) into (1), we obtain

$$\overline{M}(s)\{\gamma'(s)\dot{s}^2 + \gamma(s)\ddot{s}\} + \overline{H}(s, \dot{s}) = \overline{B}(s)\tau + \overline{\Phi}(s)^T\lambda \qquad (5)$$

where the overbar denotes transformed coefficients of (1). We can collect terms and write this as

$$\boldsymbol{a}(s)\ddot{s} + \boldsymbol{b}(s, \dot{s}) = \overline{B}(s)\tau + \overline{\Phi}(s)^T\lambda \qquad (6)$$

where $\boldsymbol{a}(s) = \overline{M}(s)\gamma(s)$ and $\boldsymbol{b}(s, \dot{s}) = \overline{M}(s)\gamma'(s)\dot{s}^2 + \overline{H}(s, \dot{s})$. Let $f(s, \dot{s})$ denote the minimum value of $\ddot{s}$ and $g(s, \dot{s})$ its maximum value, which satisfy (6) for given values of s and $\dot{s}$, and the actuator and internal force constraints (3).

We now let $\ddot{s} = u(t)$ and seek the control $u(t)$ that minimizes the traversal time t_f with $f(s, \dot{s}) \leq u \leq g(s, \dot{s})$. Given $\ddot{s} = u(t)$and the initial states $s(0) = 0$ and $\dot{s}(0) = v_i$ and the final states $s(t_f) = 1$ and $\dot{s}(t_f) = v_f$, we integrate this ordinary differential equation to obtain the desired movement $s(t)$. Q.E.D.

Theorem 1: Given a system $\ddot{s} = u(t)$ such that $f(s, \dot{s}) \leq u \leq g(s, \dot{s})$, then the time-optimal control is saturated at any instant during the motion, that is, the control is either at its maximum $u = g(s, \dot{s})$ or minimum value $u = f(s, \dot{s})$ for all $t \in [0, t_f]$.

The proof of this theorem is obtained using the Maximum Principle; see Leitmann's discussion of a time-optimal regulator with velocity dependent control bounds [7, section 13.11]. The algorithm for determining when to switch from maximum acceleration to maximum deceleration is found in [1].

IV. The Linear Programming Problem

Our result on the minimum number of saturated actuators and internal forces follows from the fact that the maximum (or minimum) acceleration of the system is the solution to a linear programming problem.

Lemma 2: If the position and velocity, s and $\dot{s}$, of the system are known, then the maximum (or minimum) value of $\ddot{s}$ is the solution to the linear programming problem: Find the value of the vector $\boldsymbol{x} \in \boldsymbol{R}^{\alpha}$ with $x_i \geq 0$, $i = \ldots, \alpha$ that maximizes (or minimizes) $\boldsymbol{c}^T\boldsymbol{x}$ subject to the β linear constraints $[A]\boldsymbol{x} = \boldsymbol{d}$.

Proof: Given a position and velocity, s and $\dot{s}$, (6) becomes a set of n linear equations in the $p+m+1$ unknowns τ, λ, and $\ddot{s}$. Assemble these unknowns into the vector $\boldsymbol{y} = (\tau, \lambda, \ddot{s})$ so $\boldsymbol{y} \in \boldsymbol{R}^{p+m+1}$. The coefficients of (6) can be assembled into the $n \times (p+m+1)$ matrix $[C] = [B, \Phi^T, -\boldsymbol{a}]$, so we have the n equality constraints on $\boldsymbol{y}$

$$[C]\boldsymbol{y} = \boldsymbol{b}. \tag{7}$$

The bounds on the actuators and internal forces are defined by (3), and for convenience, let $\ddot{s}$ be bounded $\ddot{s}^- \leq \ddot{s} \leq \ddot{s}^+$, where $\ddot{s}^-$ and $\ddot{s}^+$ are smaller and larger, respectively, than any acceleration required for the time-optimal control of this system. Therefore, we may collect the constraints into the single equation

$$\boldsymbol{y}^- \leq \boldsymbol{y} \leq \boldsymbol{y}^+. \tag{8}$$

In order to obtain the standard form for the linear programming problem, we must make a change of variables. Let the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^{p+m+1}$ be defined such that

$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{y} - \boldsymbol{y}^- \geq 0 \\ \boldsymbol{x}_2 &= \boldsymbol{y}^+ - \boldsymbol{y} \geq 0. \end{aligned} \tag{9}$$

Now define $\boldsymbol{b}_1 = \boldsymbol{b} - [C]\boldsymbol{y}^-$ so that $[C]\boldsymbol{x}_1 = [C](\boldsymbol{y} - \boldsymbol{y}^-) = \boldsymbol{b} - [C]\boldsymbol{y}^- = \boldsymbol{b}_1$, and define $\boldsymbol{b}_2 = \boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{y}^+ - \boldsymbol{y}^-$. The result is that the constraints (7) and (8) can be combined into the single equation

$$\begin{bmatrix} C & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} \quad \text{or} \quad [A]\boldsymbol{x} = \boldsymbol{d}. \tag{10}$$

Note that $\boldsymbol{I}$ is a $[p+m+1] \times [p+m+1]$ identity matrix, and $\boldsymbol{x}^T = (\boldsymbol{x}_1^T, \boldsymbol{x}_2^T)$.

The goal is to maximize $\ddot{s}$, which is the same as maximizing $\ddot{s} - \ddot{s}^-$; therefore, we define the objective function of the linear programming problem to be $\boldsymbol{c}^T\boldsymbol{x}$, where

$$\begin{aligned} \boldsymbol{c}^T &= (c_1, c_2, \ldots, c_{2(p+m+1)}), \\ c_i &= 0, \quad i \neq p+m+1, \quad c_{p+m+1} = 1. \end{aligned} \tag{11}$$

The result is that maximizing (or minimizing) the acceleration of the robot system along an admissible path becomes the linear programming problem that seeks to find the vector $\boldsymbol{x}$ with positive components by (9), which maximizes (or minimizes) the linear objective function $(\boldsymbol{c}^T\boldsymbol{x})$, subject to the linear equality constraints (10). Note that the number of unknowns is $\alpha = 2(p+m+1)$ and the number of equations is $\beta = p+m+1+n$. Q.E.D.

Assumption 1: Proper formulation of the linear programming problem requires that there be more unknowns, $\boldsymbol{x}^T = (\boldsymbol{x}_1^T, \boldsymbol{x}_2^T)$, than equations in (10); therefore, we assume that $p+m+1-n > 0$.

This assumption assures that there are enough actuators in the robotic system so that a maximum (or minimum) acceleration can be achieved.

Assumption 2: Shiller [9] shows that paths exist for which one or more of the elements of the vector $\boldsymbol{a}(s)$ in (6) may be zero over a range of motion. On these so-called *critical arcs* the desired acceleration $\ddot{s}$ is not obtained as a solution to a linear programming problem. In what follows we assume the paths contain no critical arcs.

We now state without proof some important properties of the solutions to linear programming problems.

Definition 2: The solutions to the linear constraint equations (10) that satisfy (9) form the set of *feasible solutions* to the linear programming problem.

Theorem 2 (from [3]): The objective function $(\boldsymbol{c}^T\boldsymbol{x})$ assumes its minimum (or maximum) at an extreme point of the convex set generated by the set of feasible solutions to the linear programming problem.

For our purposes, the definition of an extreme point is provided by the following theorem.

Theorem 3 (from [3]): Let the columns of the matrix $[A]$ in β linear constraints $[A]\boldsymbol{x} = \boldsymbol{d}$ be denoted P_i, $i = 1, \ldots, \alpha$ and let $\boldsymbol{P}_0 = \boldsymbol{d}$. If a set of $k \leq \beta$ vectors $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_k$ can be found that are linearly independent such that

$$x_1\boldsymbol{P}_1 + x_2\boldsymbol{P}_2 + \ldots + x_k\boldsymbol{P}_k = \boldsymbol{P}_0 \tag{12}$$

and all $x_i \geq 0$, then the point $\boldsymbol{x} = (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$ is an extreme point of the convex set of feasible solutions. Here, $\boldsymbol{x}$ is an α-dimensional vector whose last $\alpha - k$ elements are zero.

Finally we have our theorem:

Theorem 4: The time-optimal control of a robotic system, defined by n equations of the form (1) with p actuators and m differential constraints, satisfying assumptions 1 and 2, will have at least $L = p+m+1-n$ actuators or internal forces set equal to their bounds at any instant during the movement.

Proof: The matrix $[C]$ in (7) has n as its maximum rank, and the matrix $[A]$ in (10) with columns $\boldsymbol{P}_i$, $i = 1, \ldots, 2(p+m+1)$, has $n+p+m+1$ as its maximum rank, which is the maximum number of $\boldsymbol{P}_i$ that can be linearly independent. This together with Theorems 1 and 2 require that a solution x have at least $\boldsymbol{L} = 2(p+m+1) - (n+p+m+1) = p+m+1-n$ zeros. A zero value of an element of $\boldsymbol{x}$ corresponds to an actuator or an internal force equal to its bound. Recall that the bounds $\ddot{s}^+$ and $\ddot{s}^-$ are defined so that $\ddot{s}$ cannot attain them. Q.E.D.

V. Degree of Freedom

Theorem 4 applies to any dynamic system that has equations of motion of the form (1). Let us now assume that the system has a particular form.

Assumption 3: Assume the robotic system consists of N rigid bodies (including the base) and M joints connecting some or all of the bodies.

Since each of the N bodies have six degrees of freedom (except the ground body, which has zero freedom) at most $6(N-1)$ coordinates, $q \in \boldsymbol{R}^{6(N-1)}$, are required to define the configuration of the system.

Definition 3: Let the term *joint* refer to a connection between two rigid bodies that is characterized by one or more constraint equations of the form $h(q) = 0$. Constraints of this type are termed holonomic.

Definition 4: Let u_i be the number of holonomic constraint equations that characterize joint i; then $f_i = 6 - u_i$ is the *freedom* of the ith joint.

Lemma 3: We now have the well known Kutzbach–Grübler formula [6]. If the holonomic constraints imposed by the joints of the mechanism are independent, then the system of N bodies (including the base) and M joints has the degree of freedom $d = 6(N - 1 - M) + \Sigma f_i$. For a single-loop closed chain, $d = (\Sigma f_i) - 6$.

The number of coordinates n chosen to represent the configuration of a mechanism can vary in the range $d \leq n \leq 6(N-1)$. Sometimes the set of d joint parameters parameterize the configurations of the mechanism. In this case the holonomic constraints are identically satisfied and do not appear explicitly. This occurs in the case of six-degree-of-freedom (6-DOF) open-chain robots. The usual situation is that some though not all of the holonomic constraints can be eliminated in this way. For the remaining, it is more convenient to introduce them into the equations of motion as differential constraint equations. Notice that for $n > d$ coordinates, we must have $n - d$ differential constraint equations associated with the remaining holonomic constraints.

We can now divide the number of differential constraints m into k arising from nonholonomic constraints and $n - d$ obtained from the holonomic constraints. Thus, $m = n - d + k$.

Corollary to Theorem 4: For systems described by Assumption 3 with d degrees of freedom, k nonholonomic constraints, and p actuators, the minimum number of actuators and internal forces at their bounds is $L = p + k + 1 - d$.

Proof: For this case, $p + m + l - n$ becomes $p + (n - d + k) + 1 - n = p + k + 1 - d$. Q.E.D.

Example 1—A Spatial 6-DOF Robot Arm: This system has $p = 6$, $d = 6$, and $k = 0$, so it will have at least one saturated actuator.

Example 2—A 6-DOF Robot Arm with Its End-Effector Maintaining Contact with a Plane: The contact between the end-effector and the plane is a 3-DOF joint, so the system has seven joints for a total of nine joint freedoms. The system is a single-loop closed chain, so $d = 9 - 6 = 3$. With $p = 6$ and $k = 0$, we find that $L = 4$ actuators or internal forces must be saturated. If a nonholonomic constraint is applied to the movement of the end-effector on the plane, we obtain an additional saturated actuator or internal force.

Example 3—Cooperating 7-DOF Robots: The single-loop closed chain formed by two 7-DOF robots holding a workpiece has degree of freedom $d = 8$. Therefore, because $p = 14$ and $k = 0$, we have $L = 7$ saturated actuators or internal forces.

VI. Conclusion

This paper formulates the time-optimal control problem for general robotic systems and shows that the required maximum (or minimum) value of the path acceleration is the solution of a linear programming problem. The fact that such a solution is an extreme point of the set of feasible solutions allows us to determine the minimum number of actuators and internal forces that must be saturated during the time-optimal movement. The general formulation applies to a large class of robotic systems. Example calculations for several different robotic systems are provided.

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On the Global Uniform Ultimate Boundedness of a DCAL-Like Robot Controller

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Abstract—**In this paper, we illustrate how the nonadaptive part of the desired compensation adaptive law (DCAL) is a special case of a class of controllers that can be used to obtain a global stability result for the trajectory following problem for robot manipulators. This class of robot controllers is a simple linear proportional derivative (PD) controller plus additional nonlinear terms that are used to compensate for uncertain nonlinear dynamics. To analyze the stability of this class of controllers, Lyapunov's second method is used to derive a global uniform ultimate boundedness (GUUB) stability result for the tracking error. We then illustrate how the controller gains can be adjusted to obtain better tracking performance in spite of the uncertainty present in the robot manipulator dynamic equation.**

I. Introduction

Over the last decade there has been much interest in the robust tracking control of robot manipulators [3]. Tracking control means that the controller should force each joint to follow a predetermined desired joint trajectory as a function of time. As a means of quantifying how well the controller is doing its job, one usually refers to the difference between the desired joint trajectory and the actual joint trajectory as the tracking error. Therefore, if the norm of the tracking error is "small," one can be reasonably assured that the control objective is being met.

If the robot manipulator dynamics are known exactly, the so called computed-torque controller can be used to obtain good tracking performance [4]. However, an exact model of the robot manipulator is not usually available due to uncertainties such as unknown payload

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