MAE106 Homework 2 - Solution
DC Motors & Intro to the frequency domain

University of California, Irvine
Department of Mechanical and Aerospace Engineering

Problem 1

You are given the circuit shown in Figure 1. The battery outputs 6V and the potentiometer is rated at 100kΩ. Assume that the motor is not loaded and friction in the motor is negligible. Also assume that the battery and potentiometer can handle any current that the circuit needs. Show your work for all the questions below.

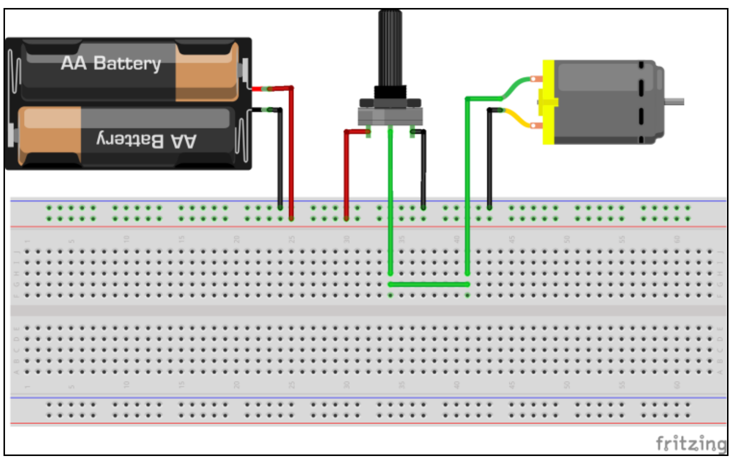


Figure 1. Circuit schematic for problem 1

1. You turn the potentiometer so that the resistance across the wiper and ground is set at 25kΩ. In this settings the spins at 120 revolutions per minute (rpm) and has an effective resistance of 5kΩ, what is the voltage across the motor?

   Based on the circuit given in Figure 1 we can assume that the voltage across the Wiper and Ground V_{WG}, with a resistance of 25kΩ, is the same as the voltage across the motor, which has given resistance of 5kΩ; these two resistances are in parallel.
   We can add these two resistances as:
   $$R^{'} = \left(\frac{1}{25k\Omega} + \frac{1}{5k\Omega}\right)^{-1} = 4.17k\Omega$$
   Now using the voltage divider equation we can find what the voltage across the motor to be:

   $$V_{out} = V_{in} * \left(\frac{R^{'}}{R_{Total}}\right) = 6V * \left(\frac{4.17k\Omega}{75k\Omega + 4.17k\Omega}\right) = 6V * 0.053 = 0.32V$$

2. You now turn the potentiometer and measure the voltage across the motor to be 5.3V. What is the resistance across the potentiometer's wiper and ground leads? What is the speed, in rpm, at which the motor is spinning?

   Again, the resistance across the motor is 5kΩ, and now the resistance across the potentiometer's wiper and ground is unknown. We can write this resistance in parallel as above:
   $$R^{'} = \left(\frac{1}{R_{wg}} + \frac{1}{R_m}\right)^{-1}$$

   $$5.3V = 6V\left(\frac{R^{'}}{(100k\Omega - R_{wg}) + \frac{1}{R^{'}}}\right) = 6V\left(\frac{1}{R'(100k\Omega - R_{wg}) + 1}\right)$$
   $$= 6V\left(\frac{1}{\frac{100k\Omega}{R_{wg}} + \frac{100k\Omega}{R_m} - 1 - \frac{R_{wg}}{R_m} + 1}\right) = 6V\left(\frac{1}{\frac{100k\Omega}{R_{wg}} + \frac{100k\Omega}{R_m} - \frac{R_{wg}}{R_m}}\right)$$

   $$1.132V = \frac{100k\Omega}{R_{wg}} + \frac{100k\Omega}{R_m} - \frac{R_{wg}}{R_m}$$

   Now solving for R_{wg}:

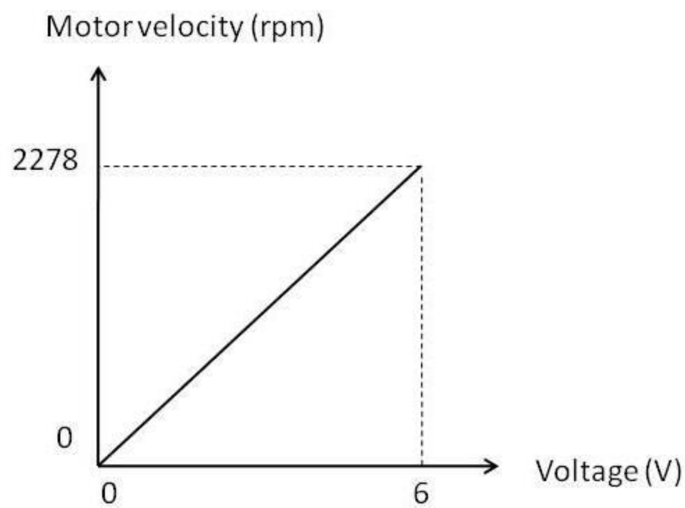
   $$\frac{1}{R_m}R_{wg} + \left(1.132V - \frac{100k\Omega}{R_m}\right) - 100k\Omega\frac{1}{R_{wg}} = 0$$

$$\frac{1}{5k\Omega}R_{wg}^2 - 18.87R_{wg} - 100k\Omega = 0$$

$$R_{wg} = \frac{18.87 \pm \sqrt{18.87^2 - 4\left(\frac{1}{5}\right)(-100)}}{\frac{2}{5}} = 5\frac{18.87 \pm \sqrt{356+80}}{2} = 5\frac{39.75}{2}$$
$$= 99.38k\Omega$$

The speed of the motor is given by the relation:
$V = B\omega$
we can solve for B because we know that at V = 0.32V the motor spins at 120rpm. Therefore:
$$\omega = V\frac{1}{B} = V * 379.71\frac{rpm}{V} = 5.3V * 379.71\frac{rpm}{V} = 2012.48rpm$$

3. Draw the motor's velocity as a function of its input voltage. Label the equation for the line that you draw. For this circuit, what is the maximum speed at which the motor can spin?



The equation of the line is:
$$Motor\ velocity = Voltage * \left(\frac{1}{0.00263}\right) = Voltage * 379.71\ \frac{rpm}{V}$$
And the maximum velocity, based on the voltage found on (2) is:
$$Motor\ velocity\_\max = 6V * 379.71\frac{rpm}{V} = 2278.27rpm$$

Problem 2

In class we studied the mathematical model of a DC brushed motor and described it as:

$$V = L\frac{di}{dt} + Ri + B\dot{\theta}$$

we then derived the torque vs. time equation for a motor that was stalled and found it to be:

$$Torque = \frac{BV}{R}(1 - e^{\frac{-t}{\tau}})$$

You have been tasked with finding a 12V DC brushed motor that can attain a steady state stall torque of $2.4N \cdot m$. This motor must also be able to reach 95% of its steady state stall torque in $20msec$ or less.

Show all your work for the questions below.

1. You find a supplier whose motors are all rated at a resistance of 100Ω. What is the minimum value of the motor's torque constant that satisfies the required stall torque?

   The steady state torque of the motor is given by:

   $$Torque_{ss} = \frac{BV}{R}$$

   from which we can solve for the motor's torque constant, B:

   $$B = \frac{R}{V} * Torque_{ss} = \frac{100\Omega}{12V} * 2.4N \cdot m = 20\frac{N \cdot m}{A}$$

2. What is the maximum value of the motor's inductance that satisfies the required response time of $20msec$? (hint: the motor needs (3*τ) seconds to reach 95% of its steady state value)

   We can first solve for the time of a single time constant, τ:

   $$3\tau = 20msec = 0.02sec$$

   $$\tau = 0.0066sec$$

   and we know that the time constant for the motor's torque curve is equivalent to:

   $$\tau = \frac{L}{R}$$

   from which we can solve for the inductance, L, with the given resistance, R=100Ω:

$L = R * \tau = 100\Omega * 0.0066s = 0.66\Omega \cdot s = 0.66H$

What is the no-load speed of this motor, in rpm, at 12V? Draw the torque vs. speed curve for this motor when it runs at 12V. Label the equation for the line that you draw.

The no-load speed is given by:

$$V = B\omega \Rightarrow \omega = \frac{V}{B} = \frac{12V}{20N \cdot m * A^{-1}} = 0.6\frac{V * A}{N \cdot m}$$

to change the units to rpms we note that:

$$V * A = Watt \qquad \textit{and} \qquad Watt = Torque * \omega = N \cdot m * \frac{rad}{sec}$$

and we therefore have that the no-load speed of the motor, in rpm, is:

$$0.6\frac{V * A}{N \cdot m} = 0.6\frac{rad}{s} \Rightarrow 0.6\frac{rad}{s} * \left(\frac{60s}{1m}\right) * \left(\frac{1rev}{2\pi}\right) = 5.73rpm$$

Problem 3

During this week's lab you used an RC circuit as a low-pass filter in order to filter the PWM signal generated with the microcontroller. The circuit is shown below:

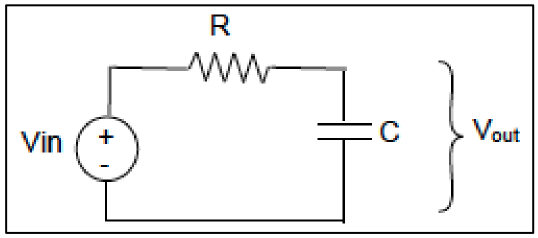


Figure 2. Circuit for problem 3

Show all your work when answering the following questions.

1. Use Kirchhoff's voltage law to find the first-order differential equation that relates the voltage across the capacitor (Vout) to Vin, R, and C.

Using KVL we have:

$$V_{in} - Ri - V_{out} = 0$$

we also know that for a capacitor we have:

$$C\frac{dv}{dt} = i$$

since the current throughout the circuit must be the same, we can write KVL as:

$$V_{in} - RC\frac{dV_{out}}{dt} - V_{out} = 0$$

solving for V_{out}:

$$RC\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

2. Using the first-order differential equation above, find the transfer function of the system. Assume all initial conditions are zero. (hint: you will need to use the Laplace transform to go from the time domain to the frequency domain).

To find the transfer function we will use the Laplace transform.

$$\mathcal{L}\left\{RC\frac{dV_{out}}{dt} + V_{out}\right\} = \mathcal{L}\{V_{in}\}$$

$$RC\big(sV_{out}(s)\big) + V_{out}(s) = V_{in}(s)$$

$$V_{out}(s)(sRC + 1) = V_{in}(s)$$

$$\frac{V_{out}(s)}{V_{in}(s)} = G(s) = \frac{1}{sRC+1} = \frac{1}{RC\left(s+\frac{1}{RC}\right)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

3. Use the transfer function derived above to find the response of the system, in the time domain, to a step change in voltage from 0 to 5V (Let R = 15 Ω and C = 0.8F). Draw the voltage vs. time curve for this step change and label the time constant and the value of the voltage at this time constant. (hint: you will need to use the inverse Laplace transform to go from the frequency domain back to the time domain; also remember that $L\{Unit\ step\ 1(t)\} = \frac{1}{s}$)).

We now have to find the output by first multiplying the transfer function by the step input:

$$V_{out}(s) = G(s)V_{in}(s) = \frac{\frac{1}{RC}}{s+\frac{1}{RC}} * \frac{5}{s} = 5\frac{\frac{1}{RC}}{s\left(s+\frac{1}{RC}\right)}$$

we can now take the inverse Laplace in order to go back to the time domain:

$$\mathcal{L}^{-1}\{V_{out}(s)\} = \mathcal{L}^{-1}\left\{5\frac{\frac{1}{RC}}{s\left(s+\frac{1}{RC}\right)}\right\}$$

If we look at the table of Laplace transformations we can find:

$$\mathcal{L}^{-1}\left\{\frac{a}{s(s+a)}\right\} = 1 - e^{-at}$$

from our equation above we can see that:

$$a = \frac{1}{RC}$$

and therefore we can find the response to a step input in the time domain as:

$$V_{out}(t) = 5\left(1 - e^{-\frac{t}{RC}}\right)$$