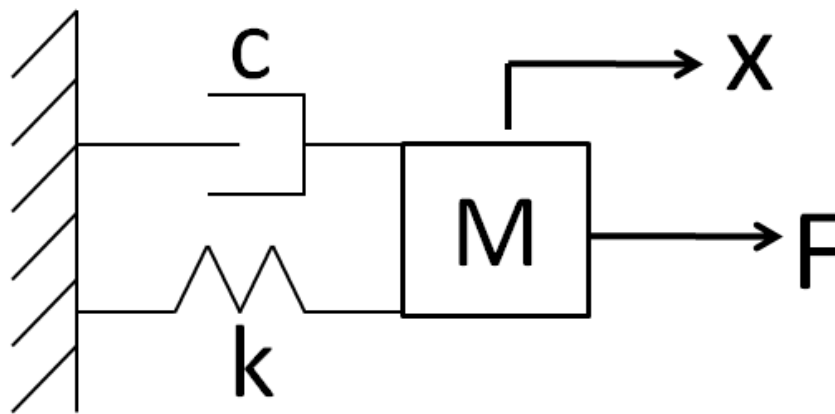# MAE106 Homework 5
Second order systems and PD control

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Problem 1

In lecture we modeled the vibrating beam as the system shown above. That is, we modeled as a mass-spring-damper system with a force input F.

1. Let $F = A\sin(\omega t)$. Find the differential equation of motion for this system.
2. Using the differential equation of motion from (1), what is the systems transfer function? (Write this expression in terms of the mass (M), damping (c), and stiffness (k) of the system).
3. Find the following values and express them in terms of the mass (M), damping (c), and stiffness (k) of the system:
   a. Natural frequency
   b. Damping ratio
   c. Critical damping
   d. Damped natural frequency
4. If we double the damping (c) of the system, how does this affect the natural frequency, damping ratio, critical damping, and damped natural frequency of the system? Express your answers in terms of the mass (M), damping (c), and stiffness (k) of the system.

Problem 2

To characterize the behavior of the mass-spring-damper system in the frequency domain we can look at the frequency response plots. For this problem, we will focus on the creating the plot for the magnitude ratio.

We can write the transfer function for the mass-spring-damper system as:

$$T(s) = \frac{kX(s)}{F(s)} = \frac{k\left(\frac{1}{m}\right)}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Note that here we multiply the numerator of the transfer function by k as this allows us to make all the following derivations in terms of ζ and ω_n.

1. Substitute s with $j\omega$ and find $T(j\omega)$.
2. Divide the numerator and denominator of $T(j\omega)$ by ω_n^2 and use the ratio of the input frequency to the natural frequency $r = \frac{\omega}{\omega_n}$ to find the transfer function in terms of r.
3. Focus on the denominator of the transfer function, what are the real and imaginary parts of the denominator?
4. What is the magnitude ratio between the numerator and the denominator in terms of ζ, ω, and ω_n? What is this value in decibels? (i.e. take the $20\log(magnitude\ ratio)$)
5. Use the expression for the magnitude ratio in decibels that you found in (4) to find the value for the following frequency ratios? Do this for the damping ratios $\zeta = 0.001, 0.25, 0.707, and\ 1$.
   a. $\frac{\omega}{\omega_n} = 0.1$
   b. $\frac{\omega}{\omega_n} = 0.5$
   c. $\frac{\omega}{\omega_n} = 1$
   d. $\frac{\omega}{\omega_n} = 5$
   e. $\frac{\omega}{\omega_n} = 10$

   *You should have 4 lines (one for each damping ratio) with 5 points each (one for each frequency ratio).*

Problem 3

Consider the mass-spring-damper system in problem 1. Assume that the mass is $10\ kg$, the damping is $0.8\ \frac{kg}{s}$, and the stiffness $0.4\ \frac{N}{m}$.

1. Compute the resonant frequency for this system. The resonant frequency is given by: $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$.

2. As the system reaches its resonance frequency, the magnitude ratio grows up to a peak value. This value is given by: $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$. What happens to the peak value if we double the damping of the system?

Problem 4

Imagine that you are driving a car along a bumpy road at $40\frac{km}{hr}$. The road's profile can be approximated by the equation: $y = 0.1\sin\left(\frac{1}{2}\pi x\right)$.
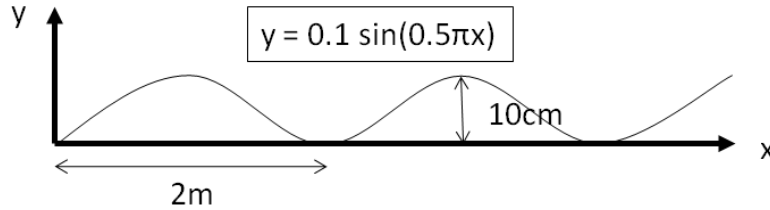

The car weighs 1500kg and its suspension has an effective stiffness of 20,000 N/m and an effective damping of 2600 N/m/s.

Your goal is to determine how much the car's body will move in the y-direction as it travels along this road.

To do this, first note that we can model the movement of the car's body as in the following diagram assuming that the car's tire contact with the road is at a single point.

To find how much the car's body will move as the car moves up and down in the road we need to find the transfer function between the road's bumpy geometry (y) and the movement of the car's body (y_{body}) (i.e. $Y_{body}(s) / Y(s)$).
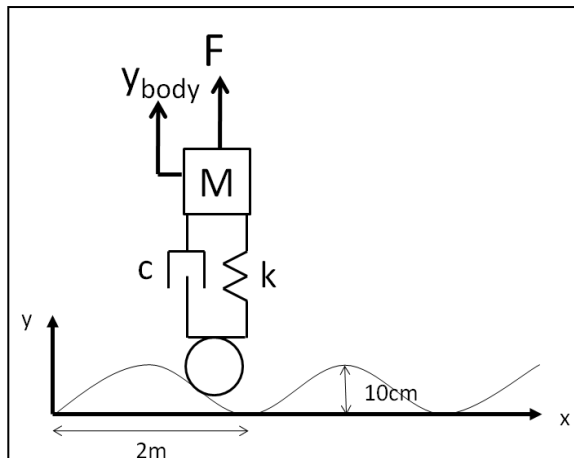
The equation of motion for the car's body is given by:

$$M\ddot{y}_{body} = c(\dot{y} - \dot{y}_{body}) + k(y - y_{body})$$

1. What is the transfer function . $Y_{body}(s) / Y(s)$?
2. How much will the car's body move (in meters) in the y direction as it travels along this road? (hint: find the magnitude ratio of the transfer function and recall that the output's magnitude equals the magnitude ratio times the input's magnitude).

The force transmitted to the car's body by the spring and damper is equal to the body's mass times its acceleration, that is: $M\ddot{y}_{body} = f_{transmitted}$. Using this relation we can derive the transfer function that relates the road's bumpy profile to the forces transmitted to the car's body as:

$$\frac{F_{transmitted}\;(s)}{Y(s)} = (cs + k)\frac{ms^2}{ms^2 + cs + k}$$

3. How much force (in N) is transmitted to the car's body. (hint: follow the same procedure as in (2)).
4. How will to the movement of the car's body in the y-direction and the force transmitted change if we double the stiffness of the suspension?