

10 Numerical Linear Algebra and Control

10.1 LMI

A Linear Matrix Inequality (LMI) has the following form:

$$F(x) = F_o + \sum_{i=1}^m x_i F_i > 0 \quad (10.1)$$

where $x \in \mathcal{R}^m$ is the unknown variables and F_i for $i = 0, 1, \dots, m$ are known symmetric matrices. This whole chapter is due to the fact that now we have efficient, easy to use, software that can solve for x , once F_i are given, basically because once the variable enters linearly or affinely, the search is convex. The software is actually available via the LMI ToolBox of Matlab.

While a course in optimization is the proper place to study this, for now all we need are a couple of facts:

- If the variables enter linearly, the search is convex
- If the search is convex, LMI tool box will find a solution if one exists

We can have matrices in (10.1), as everything will follow - as long as the unknown matrix variables enter linearly. For example, consider the search for $P > 0$, such that $-T = PA + A^T P < 0$ (for simplicity, let us do the case of dimension 2)

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

or using a notation similar to (10.1)

$$P = p_1 E_1 + p_2 E_2 + p_3 E_3 > 0 \quad (10.2)$$

where the notation is obvious. Now we can write

$$PA + A^T P = p_1(A^T E_1 + E_1 A) + p_2(A^T E_2 + E_2 A) + p_3(A^T E_3 + E_3 A) < 0$$

combining the two conditions (on P and $PA + A^T P$), we get the following

$$\sum_{i=1}^3 p_i \begin{bmatrix} -A^T E_i - E_i A & 0 \\ 0 & +E_i \end{bmatrix} > 0$$

which clearly is in the form of (10.1) - without F_0 ! Clearly, as long as you have linear appearance in the unknown matrix or scalar variables, we can do this trick. Note how we combined 2 different conditions into 1 (i.e., several LMI's in which the unknown appears linearly or affinely is still a convex problem). Fortunately, this kind of tedious manipulation is done by Matlab!

A related problem is the following

$$\text{minimize } c^T x \quad \text{subject to } F(x) > 0 \quad (10.3)$$

where c^T is the selection vector and x is the vector of unknowns. This is also a problem Matlab solves (though the 'mincx' function), though you probably can think of a way to solve it iteratively too.

The availability of LMI tool box (due to much progress in numerical linear algebra) has led to tremendous progress in control design - which is the main reason we have this chapter! Roughly speaking, we study different analysis and synthesis problem and manipulate things enough until they are in the form of either (10.1) or (10.3). Then we declare the problem solved (if there is a solution, Matlab will solve it).

10.2 Schur Complement

Let

$$A = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \quad (10.4)$$

Then, the following are equivalent:

•

$$A > 0 \iff \begin{cases} Q > 0 \\ R > 0 \\ Q - SR^{-1}S^T > 0 \end{cases}$$

•

$$A > 0 \iff \begin{cases} Q > 0 \\ R > 0 \\ R - S^T Q^{-1}S > 0 \end{cases}$$

Proof: Clearly, $A > 0$ implies $Q > 0$ and $R > 0$ (use $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ y \end{pmatrix}$ in the basic definition of positive definiteness). So we concentrate on the third property; i.e., given $Q > 0$ and $R > 0$, then $A > 0$ is equivalent to the third property. Obviously, we only need to do the first form (and the second follows closely).

We start with a generic vector $z = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$z^T A z = x^T Q x + y^T R y + x^T S y + y^T S^T x > 0 \quad \forall x, y \quad (10.5)$$

To show \Rightarrow , we use *any* $x \neq 0$ along with $y = -R^{-1}S^T x$ to get

$$\begin{aligned} z^T A z &= x^T Q x + x^T S R^{-1} R R^{-1} S^T x - x^T S R^{-1} S^T x - x^T S R^{-1} S^T x \\ &= x^T (Q - S R^{-1} S^T) x > 0 \quad \forall x \neq 0 \end{aligned}$$

thus $A > 0$ implies that $(Q - SR^{-1}S^T) > 0$.

To show \Leftarrow , we go back to equation (10.5) : Add and subtract a $x^T SR^{-1}S^T x$, to get :

$$\begin{aligned} z^T Az &= x^T QX - x^T SR^{-1}S^T x + x^T SR^{-1}S^T x + y^T Ry + x^T Sy + y^T S^T x \\ &= x^T (Q - SR^{-1}S^T)x + (y^T + x^T SR^{-1}) R (y + R^{-1}S^T x) \end{aligned}$$

for all x and y . The first term is strictly positive and the second term is in the form of $B^T RB$ with $R > 0$, so it is as worst zero. Therefore $z^T Az > 0 \quad \forall z$. \square

What if we had $A \geq 0$? The only modification would have been the following:

- If $R > 0$, then

$$A \geq 0 \iff \begin{cases} Q \geq 0 \\ Q - SR^{-1}S^T > 0 \end{cases} \quad (10.6)$$

- If $Q > 0$, then

$$A > 0 \iff \begin{cases} R \geq 0 \\ R - S^T Q^{-1} S > 0 \end{cases} \quad (10.7)$$

Naturally, everything above follows if we replace all ' $>$ ' signs with ' $<$ ' signs. That is

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$$A < 0 \iff \begin{cases} Q < 0 \\ R < 0 \\ Q - SR^{-1}S^T < 0 \end{cases} \quad (10.8)$$

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$$A < 0 \iff \begin{cases} Q < 0 \\ R < 0 \\ R - S^T Q^{-1} S < 0 \end{cases} \quad (10.9)$$

Finally, we have the following: IF

$$C = \begin{pmatrix} Q & S \\ S^T & 0 \end{pmatrix} \geq 0 \quad (\text{or } \leq 0) \quad \text{then } S = 0 \quad (10.10)$$

(pick any $z^T = (x^T \ y^T)$, show that there exists another vector $\hat{z}^T = (x^T \ \hat{y}^T)$ such that the sign of $z^T C z$ is opposite of $\hat{z}^T C \hat{z}$ which is impossible. The only way out is to have $x^T S y = 0$ for all x and y which is mean $S = 0$)

10.3 S-Procedure

Let T_o, T_1, \dots, T_p be symmetric matrices. If there exists a set of positive scalars $\tau_i > 0$ such that

$$T_o - \sum_1^p \tau_i T_i > 0 \quad (10.11)$$

then the following is true

$$x^T T_o x > 0 \text{ for all } x \neq 0 \text{ such that } x^T T_i x > 0 \forall i \quad (10.12)$$

Clearly, (10.11) implies (10.12) - to see this just multiply (10.11) by x^T and x . We end up using it as a sufficient condition - possibly conservative - by finding τ_i such that (10.11) holds. When $p = 1$, this is sufficient and necessary, but in general there is some conservatism associated with it.

We often use negative conditions, which are obtained by using $S_i = -T_i$: Let S_o, S_1, \dots, S_p be symmetric matrices. If there exists a set of positive scalars $\tau_i > 0$ such that

$$S_o - \sum_1^p \tau_i S_i < 0 \quad (10.13)$$

then the following is true

$$x^T S_o x < 0 \text{ for all } x \neq 0 \text{ such that } x^T S_i x < 0 \forall i \quad (10.14)$$

10.4 Ellipsoids

We will use the concept of ellipsoidal sets a great deal. Simply, consider the following: Given $P > 0$, we define the following ellipsoid:

$$\mathcal{E}(P, c) = \{x : x^T P x \leq c\} \quad (10.15)$$

Naturally, we need $P > 0$ if we want a bounded set. Let us consider some of its basic properties:

- $x \in \mathcal{E}(P, c) \Leftrightarrow \alpha x \in \mathcal{E}(P, c\alpha^2)$
- $c_1 \leq c_2 \Leftrightarrow \mathcal{E}(P, c_1) \subseteq \mathcal{E}(P, c_2)$
- If $P_1 \leq P_2 \Leftrightarrow \mathcal{E}(P_2, c) \subseteq \mathcal{E}(P_1, c)$
- ellipsoid not centered at the origin:

$$\mathcal{E}(P, x_o, c) = \{x : (x - x_o)^T P (x - x_o) \leq c\}$$

- The volume of an ellipse is $\alpha_n(\det P^{-1})^{\frac{1}{2}}$, where α is the volume of the n -dimensional unit 2-ball; i.e.,

$$\alpha_n = \frac{\pi^{n/2}}{(n/2)!} \text{ if } n \text{ is even}$$

$$\alpha_n = \frac{2^n \pi^{(n-1)/2} ((n-1)/2)!}{n!} \text{ if } n \text{ is odd}$$

but an upper bound (thus only a potentially conservative estimate) to it is more suitable for matlab use. For this, recall (??)

$$(\det P)^{1/n} \leq \frac{\text{trace}(P)}{n}$$

which implies that to minimize volume (thus $\det(P)$ - which is very difficult), we can try to minimize $\text{trace}(P^{-1})$ or at times $\text{trace}(Q)$ where $Q > P^{-1}$.

10.4.1 Norm of a vector in an ellipsoid

One of the main tricks we will be using, over and over, is to find the max of $\|y = Cx\|$ for all $x \in \mathcal{E}(P, c)$ (what is the minimum value?). Consider the following

$$\begin{pmatrix} P & C^T \\ C & \frac{\gamma^2}{c} I \end{pmatrix} > 0 \Leftrightarrow P - \frac{c}{\gamma^2} C^T C > 0 \quad (10.16)$$

Therefore, if this condition (i.e., (10.16)) holds, we have

$$\|y\| \leq \gamma$$

10.5 Stability, L_2 gain, etc

We now start with the analysis results; i.e., conditions that can be checked relatively easily and would imply stability, finite gain, etc.

10.5.1 stability

Suppose we have the following dynamics

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_o \end{cases} \quad (10.17)$$

One way of checking the stability of A was through Lyapunov method and Lyapunov equation: finding the solution

$$PA + A^T P = -Q, \quad Q > 0$$

which means we must have $P > 0$ such that $PA + A^T P < 0$. This is a simple search (for P) which is convex, this easily done with Matlab LMI software.

10.5.2 H_∞ or L_2 gain

Suppose we have the following dynamics

$$\begin{cases} \dot{x} &= Ax + B_1 w \\ y &= Cx + Dw \\ x(0) &= 0 \end{cases} \quad (10.18)$$

We want to find the L_2 or energy gain of the system, ie., the smallest γ such that

$$\int_0^\infty y^T y \, dt \leq \gamma^2 \int_0^\infty w^T w \, dt$$

(this also becomes the famous H_∞ problems). The basic approach is to use a Lyapunov like function

$$V(x) = x^T(t)Px(t)$$

where $P > 0$, and find the smallest γ that satisfies:

$$\mathcal{T} = \dot{V}(t) + y^T y - \gamma^2 w^T w \leq 0 \quad (10.19)$$

since simply integrating both sides from zero to infinity gives the energy (or L_2) gain. Note that this gives an estimate (or upper bound) on the actual gain and there may be some conservatism.

To get a more tractable form of (10.19), we simply take derivative of V and substitute:

$$\begin{aligned} \mathcal{T} = \dot{V}(t) + y^T y - \gamma^2 w^T w &= x^T PAx + x^T A^T Px + x^T PBw + w^T B^T Px \\ &\quad (Cx + Dw)^T (Cx + Dw) - \gamma^2 w^T w \end{aligned}$$

which can be written as

$$\mathcal{T} = \begin{pmatrix} x^T & w^T \end{pmatrix} \begin{pmatrix} PA + A^T P + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

A sufficient condition for (10.19) is this the following

$$\begin{pmatrix} PA + A^T P + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{pmatrix} < 0 \quad (10.20)$$

which is the same as the following (doing the Schur complement)

$$\begin{pmatrix} PA + A^T P & PB \\ B^T P & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} C^T \\ D^T \end{pmatrix} I \begin{pmatrix} C & D \end{pmatrix} < 0$$

but using Schur complement formula, this is equivalent to

$$\begin{pmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma^2 I & D^T \\ C & D & -I \end{pmatrix} < 0. \quad (10.21)$$

A sequence of pre and post multiplying by $\begin{pmatrix} 1/\sqrt{\gamma} & 0 & 0 \\ 0 & 1/\sqrt{\gamma} & 0 \\ 0 & 0 & \sqrt{\gamma} \end{pmatrix}$, and using $\hat{P} = P/\gamma$ yields

$$\begin{pmatrix} \hat{P}A + A^T \hat{P} & \hat{P}B & C^T \\ B^T \hat{P} & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0. \quad (10.22)$$

There is another - equivalent - form that is used in the synthesis problems. That form is obtained by pre and post multiplying (10.22) for example, by $\begin{pmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ where $Q = \hat{P}^{-1}$, results in

$$\begin{pmatrix} AQ + QA^T & B & QC^T \\ B^T & -\gamma I & D^T \\ CQ & D & -\gamma I \end{pmatrix} < 0. \quad (10.23)$$

Often we see references to Bounded Real Inequality, which could be any of the above. In particular, (10.20) can be written (doing a Schur complement) as

$$PA + A^T P + C^T C + (PB + C^T D)(\gamma^2 - D^T D)^{-1}(B^T P + D^T C) < 0 \quad (10.24)$$

or if $D = 0$, the simple form of

$$PA + A^T P + C^T C + \frac{1}{\gamma^2} PBB^T P < 0. \quad (10.25)$$

In all of these problems (i.e., (10.20)-(10.23)), we seek a positive definite matrix (P or Q) so that the linear matrix inequality - in which the variable P enters linearly- is satisfied. This is a standard convex search and LMI toolbox can be used to solve for it. Furthermore, we would like to minimize γ . Again, this ends up being a convex problem (generalized eigenvalue problem) for which matlab can be used easily (if nothing else, think of matlab solving this minimization problem as a sequence of feasibility problems with decreasing values of γ which should converge to any tolerance in finite steps).

10.5.3 Invariant/Reachable sets

Consider the system in (10.18). Let us try to find some estimates for the reachable set or the invariant set. The reachable set (from zer) is the set of points the state vector can reach with zero initial condition, given some limitations on the disturbance. The invariant set is a similar (but not identical necessarily) concept. The invariant set is the set that the state vector does not leave once it is inside of it, again given some limits on the disturbance. (which one of then is stronger, so to speak, why?). Generally, these are hard to characterize (as you might have seen in the exam before!).

We discuss both types, depending on the form of the disturbance bound. In all case, we use a Lyapunov function and an ellipsoidal sets for this estimate; i.e., our estimates are in the following form

$$\mathcal{E} = \{x : V(x) = x^T P x \leq c\}$$

Obviously, there are varying degrees of conservatism associated with these, but as you will see, they are easy to obtain and can be applied to nonlinear systems as well.

Case A:Energy Bounded Disturbance:

Suppose the disturbance w had a energy bound;

$$\int_0^\infty w^T w dt \leq w_{max}^2$$

then if we can find P such than

$$\dot{V} - w^T w \leq 0 \tag{10.26}$$

then the reachable set is $V(x) \leq w_{max}^2$. To see this, simply integrate (10.26) from zero to t

$$V(t) - V(0) - \int_0^t w^T w dt \leq 0$$

With zero initial conditions, we have

$$V(x) = x^T P x \leq \int_0^t w^T w dt < w_{max}^2$$

which can also be refined as

$$\|x\|^2 < \frac{1}{\lambda_{min} P} w_{max}^2.$$

In any case, to find the V or P , we expand the \dot{V} term in (10.26); i.e., we seek P such than the following holds

$$x^T P A x + x^T A^T P x + x^T P B w + w^T B^T P x - w^T w < 0$$

where this is the same as

$$(x^T \ w^T) \begin{pmatrix} P A + A^T P & P B \\ B^T P & -I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} < 0$$

As a result, we have the following estimate (sufficient condition): Suppose there is $P > 0$ satisfying

$$\begin{pmatrix} P A + A^T P & P B \\ B^T P & -I \end{pmatrix} < 0 \tag{10.27}$$

then an estimate of the reachable set is

$$\mathcal{E} = \{x : x^T P x \leq w_{max}^2\}$$

Case B: Peak Bounded Disturbance:

Suppose we have a bound on the peak norm of the disturbance:

$$w^T(t)w(t) \leq w_{max}^2 \quad \forall t$$

Note that if you are interested in element wise peak norm, you might be off by a factor of \sqrt{m} where m is the dimension of w . Here the basic condition is the following: Suppose there was a P for a $V = x^T P x$ such that

$$\dot{V} + \alpha(V - w^T w) < 0 \tag{10.28}$$

for some $\alpha > 0$. Then $V \leq w_{max}^2$ is an attractive invariant set (i.e., if you start from inside, you never leave, and if you start from outside, you get attracted to it!). It is not that important perhaps, but α come from the S-procedure (setting the problem up as having $\dot{V} < 0$ where $V > w^T w$). It is relatively easy to see that (10.28) implies that

$$\dot{V} + \alpha(V - w_{max}^2) < 0$$

which implies that if you are insider of $V \leq w_{max}^2$ you cannot go outside and if you are outside, V gets smaller - until you get inside! Going through the same calculations as before we get the following conditions for P :

$$\begin{pmatrix} PA + A^T P + \alpha P & PB \\ B^T P & -\alpha I \end{pmatrix} < 0 \tag{10.29}$$

then an estimate of the reachable (and/or invariant) set is

$$\mathcal{E} = \{x : x^T P x \leq w_{max}^2\}$$

Note that in this case, the term αP is nonlinear in unknown variables, which destroys the convexity. Generally, a simple line search (ie, iterative) is done on α . This is not a major problem, since the (1,1) block of the inequality above shows that α is between zero and half of the real part of the least stable eigenvalue of A (why?).

10.5.4 Energy to peak and peak to peak gains

To obtain (upper) bounds for the energy-to-peak or peak-to-peak gains for a system, we simply combine the previous subsection results with those regarding the norm of a vector in the ellipsoid:

- **Energy to Peak norm:** Suppose (10.27) and (10.16) hold for some P , $\gamma = \gamma^*$ and $c = 1$, then it is easy to see that $x^T P x \leq 1$ is the reachable set, as long as $\|w\|_{L_2} \leq 1$. Then, (10.16) implies that the norm of y is less than γ^* in this ellipsoid.

- **Peak to Peak norm:** Suppose (10.29) and (10.16) hold for some P , $\alpha > 0$, $\gamma = \gamma^*$ and $c = 1$, then it is easy to see that $x^T P x \leq 1$ is the reachable set, as long as $\|w(t)\| \leq 1$. Then, (10.16) implies that the norm of y is less than γ^* in this ellipsoid.

As a result, we solve for $P > 0$, such that the LMI's holds, while minimizing γ^* .

10.6 Synthesis

Suppose our system is in the following form

$$\dot{x} = Ax + B_1 w + B_2 u \quad (10.30)$$

$$z = C_1 x + D_{11} w + D_{12} u \quad (10.31)$$

$$y = C_2 x + D_{21} w + D_{22} u \quad (10.32)$$

where w is the disturbance and u is the control input. Vector z is called controlled output, and contains states or combinations that we want to penalized or reduce, etc (which may be the same or different from the measured output y). Almost always, we use $D_{22} = 0$, to simplify things (well posed problem issue). Often it is justified through a simple transformation of the form $\hat{y} = y - D_{22}u$ - as long as u is available.

Throughout this long subsection, we will try the synthesis problem: i.e, finding a control law - with different structures - such that the closed loop has desirable properties (e.g., stability, small L_2 gain, small peak to peak gain, etc).

10.6.1 State Feedback Controllers

Suppose you are designing a state feedback controller of the form

$$u = Kx \quad (10.33)$$

Putting it back in the original equation of motion (i.e., (10.30)), yields the following closed loop equation:

$$\dot{x} = (A + B_2 K)x + B_1 w \quad (10.34)$$

$$z = (C_1 + D_{12} K)x + D_{11} w \quad (10.35)$$

We try to do the problem of stabilizing controller only; i.e., when $w = 0$. Other considerations, such as minimizing the L_2 gain of the closed loop or the energy to peak or peak to peak gains are quite similar, and left as exercise.

We start by hoping to find a matrix P such that a Lyapunov function candidate of the of form

$$V(x) = x^T P x$$

can do the trick. For this $V(x)$ to work, we need $\dot{V}(x) < 0$; i.e.

$$P(A + B_2K) + (A + B_2K)^T P = PA + PB_2K + A^T P + K^T B_2^T P < 0 \quad (10.36)$$

The problem is the PB_2K term, which is nonlinear in the unknown variables (i.e., not a linear matrix inequality, thus not a convex search and no LMI-toolbox!). Fortunately, there is a little trick that solves this dilemma. Use

$$W = KP^{-1} = KX \quad (10.37)$$

and note that pre- and post multiplying (10.36) by $X = P^{-1}$ we get the following sufficient condition

$$AX + XA^T + B_2W + W^T B_2^T < 0 \quad (10.38)$$

which is now linear in X and W . So we use LMI-toolbox (or any similar program) to find W and $X > 0$ that satisfy (10.38). Then the control law (from (10.37)) is

$$K = WX^{-1}.$$

10.6.2 Output feedback design

Now suppose we do not have access to all of states and instead had access to :

$$\begin{cases} \dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w \end{cases} \quad (10.39)$$

Where, as before, we have assumed that there is no feed-through term from u to y . We need to design a compensator of the form:

$$\begin{cases} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c \end{cases} \quad (10.40)$$

Combining the two, we get the closed loop dynamics of

$$\begin{cases} \dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w \\ z &= C_{cl} x_{cl} + D_{cl} w \end{cases} \quad (10.41)$$

where $x_{cl}^T = (x^T \ x_c^T)$ and

$$A_{cl} = \begin{pmatrix} A & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix}, \quad B_{cl} = \begin{pmatrix} B_1 \\ B_c D_{21} \end{pmatrix}$$

and

$$C_{cl} = (C_1 \ D_{12} C_c) \quad , \quad D_{cl} = D_{11}$$

Now let us discuss stability only (L_2 and stuff follows pretty similar to this) - so we set $w = 0$. For stability, it is sufficient to have $P > 0$ such that

$$PA_{cl} + A_{cl}^T P < 0$$

or using $Q = P^{-1}$, if we could find $Q > 0$ such that

$$A_c Q + Q A_c^T < 0 \quad (10.42)$$

where the dimension of Q is $2n \times 2n$; i.e, has a structure

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{pmatrix} \quad (10.43)$$

Now we can say, without any loss of generality that Q_2 is non-singular (one can always add a little bit to it; e.g., $Q_2 = Q_2 + \epsilon I$ so that the off-diagonal terms is non-singular without changing $Q > 0$ or the overall inequality in (10.42)). Once this is done, we can show that any matrix in the form in (10.43) - with Q_2 nonsingular, can be transformed to the following:

$$TQT^T = \begin{pmatrix} X & X \\ X & S^{-1} + X \end{pmatrix}, T = \text{diag}\{I, T_1\}$$

for some appropriately defined T_1 . Naturally, T_1 , X and S are functions of Q_i ! Furthermore, it is relatively easy to show that this transformation only changes the ‘realization’ of the compensator and nothing else. In summary, we can - without any loss of generality, say that if (10.42) has a solution Q , it has the form

$$Q = \begin{pmatrix} X & X \\ X & S^{-1} + X \end{pmatrix} \quad (10.44)$$

Next, we define

$$Y = S + X^{-1} \implies \begin{pmatrix} X & X \\ X & S^{-1} + X \end{pmatrix}^{-1} = \begin{pmatrix} Y & -S \\ -S & S \end{pmatrix}$$

and go back to (10.42) and pre and post multiplying it (i.e., congruent transformation) by T_2 and T_3 , respectively, where

$$T_2 = \begin{pmatrix} T_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad T_3 = \begin{pmatrix} Y & -S \\ I & 0 \end{pmatrix}$$

After a good bit of manipulations, we get the following - from (10.42):

$$\begin{pmatrix} YA + A^T Y - SB_c C_2 - C_2^T B_c^T S & Y[A + B_2 C_c]X - S[A_c + B_c C_2]X + A^T \\ * & AX + XA^T + B_2 C_c X + X C_c^T B_2^T \end{pmatrix} < 0 \quad (10.45)$$

The inequality above does not ‘look’ linear, but if we use the following

$$W_c = C_c X$$

$$W_o = -SB_c$$

$$L = Y[A + B_2 C_c]X - S[A_c + B_c C_2]X = YAX + YB_2 W_c - SA_c X + W_o C_2 X$$

we get

$$\begin{pmatrix} YA + A^T Y + W_o C_2 + C_2^T W_o^T & L + A^T \\ L^T + A & AX + X A^T + B_2 W_c + W_c^T B_2^T \end{pmatrix} < 0 \quad (10.46)$$

which is linear in X, Y, W_o, W_c and L . Once these are found (by LIM-toolbox!) we get the compensator from the following

$$C_c = W_c X^{-1} \quad (10.47)$$

$$B_c = -S^{-1} W_o \quad (10.48)$$

$$A_c = S^{-1} (-L X^{-1} + Y A + Y B_2 C_c) - B_c C_2 \quad (10.49)$$

Actually, X and Y should be such $S = Y - X^{-1} > 0$ - which through Schur is equivalent to

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0 \quad (10.50)$$

In summary, the compensator is obtained by searching for the unknown variables that satisfy (10.46) and (10.50) (and then A_c etc from (10.47))

10.7 Multi-objective problems

The basic idea of multi-objective approach is to design a controller such that two (or more) different objectives are met. Consider the state feedback problem we discussed earlier, the closed loop is

$$\dot{x} = (A + B_2 K)x + B_1 w \quad (10.51)$$

$$z = (C_1 + D_{12} K)x \quad (10.52)$$

where we have set $D_{11} = 0$. So suppose we needed to find K such that the L_2 gain from w to z was less than γ_2 while minimizing the energy to peak gain γ^* for disturbances with unit energy- again from w to z - to have bounded energy in z while minimizing peak. Following the development of earlier sections, these objectives will be satisfied if

$$\begin{pmatrix} \hat{P} A_{cl} + A_{cl}^T \hat{P} & \hat{P} B_{cl} & C_{cl}^T \\ B_{cl}^T \hat{P} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{pmatrix} < 0 \quad (10.53)$$

for the desired γ_2 , while the second objective is to minimize γ^* in

$$\begin{pmatrix} P A_{cl} + A_{cl}^T P & P B_{cl} \\ B_{cl}^T P & -I \end{pmatrix} < 0 \quad (10.54)$$

$$\begin{pmatrix} P & C_{cl}^T \\ C_{cl} & (\gamma^*)^2 I \end{pmatrix} > 0 \quad (10.55)$$

Recall that in this case, an estimate of the reachable set is

$$\mathcal{E} = \{x : x^T P x \leq w_{max}^2 = 1\}$$

Now the most general result would be that the two objectives would be met by two different P , one for each objective, but as you recall each problem will result in a search for P and $W = KP^{-1}$. Since we want only one controller, we need to have $K_1 = W_1 P_1^{-1} = W_2 P_2^{-1}$ which kills any convexity we might have!

What is done is to use the same P in both objectives (which means the same W). This allows us to solve the problem but can be quite conservative. Dealing with this conservatism is an active area of research.

10.8 Time variations and mild nonlinearities

One of the advantages of using Lyapunov functions, as mentioned earlier, is that the basic approach (including reliance on LMI) can be extended to certain class of time varying and/or nonlinear systems - with relative ease. We will do a little bit of review of these things.

First, however, we need to look into the concept of **Quadratic Stability**. If a dynamical system is shown to have Lyapunov function V that is quadratic in x such that $\dot{V} < -\epsilon \|x\|^2$, then the system is quadratically stable (**QS**) - which is stronger than asymptotic stability (e.g., we can show exponential decay). The simplest form of V - you guessed it - for this method is $V = x^T P x$ for some $P > 0$. As you recall, for linear time invariant (LTI) systems this was equivalent to traditional stability (checked with eigenvalues), except quadratic stability can be extended to more general systems.

10.8.1 Robustness and quadratic stability

Consider the simple mass, spring dashpot model

$$m\ddot{y} + c\dot{y} + ky = u$$

or

$$\begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu$$

If stiffness or damping were unknown (either constant or slowly changing). Then we can start with $V(x) = x^T P x$ and establish stability (e.g., when $u = 0$), or L_2 gain (when u is disturbance) or even design control. For then we typically mean to find P satisfying

$$PA(q) + A(q)^T P < 0$$

where $A(q)$ denotes dependence of A on the unknown parameter q - for example $k = k_{nom} + q$. Now this last inequality still forms an LMI as long as dependence

of $A(q)$ on q is linear. In particular, obtaining a P satisfying it follows if we had a P satisfying

$$PA(q_{min}) + A(q_{min})^T P < 0 \quad \text{and} \quad PA(q_{max}) + A(q_{max})^T P < 0$$

These lead to sufficient conditions for robust stability, but they could be conservative. For LTI systems, the famous μ -synthesis leads to much less conservatism, but QS method also applies if the uncertainty is time varying (though how fast is another story!)

10.8.2 Linear parameter varying systems

Linear parameter varying systems are those systems whose model is time varying, but linear and the variation is keys to a (or few) specific parameter - which is supposed to be measured on-line. There is a lot of similarities to gain scheduling (which is a whole new can of worms). A particularly interesting form of this is the quasi-lpv. For example, consider the simple pendulum model

$$\ddot{\theta} + \sin \theta = u$$

or in state space form (with $x^T = (\theta \ \dot{\theta})^T$), or

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{\sin x_1}{x_1} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = A(\rho)x + Bu$$

where $\rho(x) = \frac{\sin x_1}{x_1}$, which leads to $0 \leq \rho \leq 1$. Now, if we measure $x_1 = \theta$, then we have $\rho(t)$ on-line! Note that model here has a lot of similarities with the robust problem discussed above, with one difference: in robust problem we do not know the values of q - ever! In the q-lpv (or lpv) however, knowing the parameter can help us do a better job. For example, suppose you had limited torque for the pendulum. If you only had one K , this had to be chosen so that no matter what x , $Kx \leq u_{lim}$. However, you could different K 's, as θ gets smaller, you would use a larger K and thus can be more aggressive!

The approach - i.e, finding $K(\rho)$ - is quite similar to gain scheduling and at times is called self-scheduling (it is scheduled based on its own response! and not external command). This is an area of research that has been quite active in recent years.

10.9 Exercises

1. Show the first three properties of ellipsoids are true
2. Show (10.16) actually does bound the norm of y
3. Show that $\gamma^T - D^T D > 0 \Leftrightarrow \sigma_{max} D < \gamma$
4. Verify the equivalency of (10.21) and (10.22)
5. Consider the invariant set for the peak bounded disturbance (e.g., (10.29)). Show that if x is not inside this set, it will reach it (i.e., contractive). Estimate the rate of convergence!
6. For the peak bounded case, use S-procedure to come up with matrix inequality for the norm of $y = Cx + Dw$.
7. Beyond stability: How would you modify the Lyapunov inequality if we needed $x(t)$ to decay as least as fast as $e^{-\alpha t}$?
8. Do the estimate for peak to peak and energy to peak if the appropriate norm of $w(t)$ was bounded by w_{max} instead of 1.
9. In (10.30), what would be z if we wanted to penalize absolute acceleration of a single mass-spring dashpot plus an actuator?
10. In the state feedback problem, do the L_2 problem: Find $u = Kx$ such that the closed loop L_2 gain is minimized.
11. In the output feedback case, show the details needed to establish that the special structure of Q is without loss of generality - including the change of controller representation.
12. In the output feedback case, do the L_2 gain minimization problem !
13. Filling some intermediate steps: mincx vs trial and error: solve the minimum γ in one of the minimization problem by 'mincx' function and then get the same by checking feasibility with decreasing values of γ
14. **Important:** In the multi-objective problem, solve the following: Given w with energy w_{max} , minimize the L_2 gain subject to the saturation bound u_{lim}