4 THE LINEAR QUADRATIC REGULATOR

In this Section, we will deal with the 'Linear Quadratic Regulator' problem (or LQR for short). We start with the most general from; that of time varying system matrices and finite horizon.

4.1 Time varying and finite horizon case

Consider the dynamical system

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ x(t_o) = x_o \end{cases} \tag{4.1}$$

We are interested in finding a control $u(t)$ such that the following cost functional is minimized

$$J(t_o, t_f, x_o, u(.)\,) = \int_{t_o}^{t_f} \{\, x^T(t)Q(t)x(t) + u^T(t)R(t)u(t),\, \} \, dt \; + \; x^T(t_f)\,P_1\,x(t_f) \tag{4.2}$$

where

$$\begin{cases} t_f \; is \; the \; fixed \; final \; time \\ P_1 \geq 0 \; is \; the \; terminal \; penalty \; term \\ Q(t) \geq 0 \; \forall t \in [0, t_f] \\ R(t) > 0 \; \forall t \in [0, t_f] \end{cases} \tag{4.3}$$

The desired solution would give us a control law for $u(.)$. This may or may not be feedback (or even linear). While there are several ways to approach this problem, we will use the perturbation or variation approach. For this, we make the following *assumption*:

Assumption 4.1. *Suppose there exists an optimal control law $u^*(.)$ that minimizes* (4.2)*, subject to* (4.1)*.*

Therefore, any other control law cannot do better! Now let us implement $u^*(t)$, and label the resulting trajectory, which minimizes (4.2), $x^*(t)$; i.e.,

$$\begin{cases} \dot{x}^*(t) = A(t)\,x^*(t) + B(t)\,u^*(t) \\ x^*(t_o) = x_o \end{cases} \tag{4.4}$$

Now *all other* control laws can be represented by

$$u(t) = u^*(t) + \epsilon\,\tilde{u}(t), \quad t \in [0, t_f] \tag{4.5}$$

where ϵ is a (possibly negative) scalar and $\tilde{u}(t)$ is the control perturbation (function of time). Note that if $u(t)$ is implemented, the resulting trajectory will be the $x(t)$ of (4.1).

At this point we can introduce the state perturbation $\tilde{x}(t)$

$$\epsilon\tilde{x}(t) \stackrel{\triangle}{=} x(t) - x^*(t) \iff x(t) = x^*(t) + \epsilon\tilde{x}(t) \tag{4.6}$$

where ϵ is the same as in (4.5) and $x(t)$ is the response to the control $u(t)$ - as in (4.5). Combining (4.5) and (4.6), we get

$$\begin{cases} \dot{\tilde{x}}(t) = A(t)\,\tilde{x}(t) + B(t)\,\tilde{u}(t) \\ \tilde{x}(t_o) = 0. \end{cases} \tag{4.7}$$

From the first class in linear systems, the solution (or response) to (4.7) is

$$\tilde{x}(t) = \int_{t_o}^{t} \Phi(t,\tau)\,B(\tau)\tilde{u}(\tau)\,d\tau \tag{4.8}$$

where $\Phi(t,\tau)$ is the state transition matrix associated with $A(t)$. Recall that state transition matrix satisfies the following

$$\begin{cases} \Phi^{-1}(t,\tau) = \Phi(\tau,t) \\ \Phi(t,t) = I \\ \Phi(t,t_1)\,\Phi(t_1,t_2) = \Phi(t,t_2) \quad \forall t_1 \in [t,t_2] \\ \frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau) \end{cases} \tag{4.9}$$

Let us go back to our problem. Since $u^*(.)$ is the optimal control, no ϵ or $\tilde{u}(.)$ can result in a smaller cost function than the following optimal one

$$\begin{aligned} J_{min} &= J(t_o, t_f, x_o, u^*(.)) \tag{4.10} \\ &= \int_{t_o}^{t_f} \{ x^{*T}(t)Q(t)x^*(t) + u^{*T}(t)R(t)u^*(t) \}\,dt + x^{*T}(t_f)\,P_1\,x^*(t_f). \end{aligned}$$

Therefore, if we calculate the cost due to a non-optimal control law, we expect to have a larger than (or at best equal to) J_{min}. That is, if we implement some $u(t) = u^*(t) + \epsilon\,\tilde{u}(t)$, the cost will be

$$\begin{aligned} J(t_o, t_f, x_o, u(.)) &= \int_{t_0}^{t_f} \{ [x^*(t) + \epsilon\tilde{x}(t)]^T Q(t)[x^*(t) + \epsilon\tilde{x}(t)] \}\,dt \\ &+ \int_{t_0}^{t_f} \{ [u^*(t) + \epsilon\tilde{u}(t)]^T R(t)[u^*(t) + \epsilon\tilde{u}(t)] \}\,dt \\ &+ [x^*(t_f) + \epsilon\tilde{x}(t_f)]^T P_1 [x^*(t_f) + \epsilon\tilde{x}(t_f)] \tag{4.11} \end{aligned}$$

which cannot be any less that J_{min}. Now let us rearrange (4.11) and group in terms of powers of ϵ.

$$J(t_o, t_f, x_o, u(.))$$

$$= \int_{t_0}^{t_f} \{\, x^{*T}(t)Q(t)x^*(t) + u^{*T}(t)R(t)u^*(t)\,\}\, dt \;+\; x^{*T}(t_f)\, P_1\, x^*(t_f)$$

$$+ \;\; 2\epsilon \left[\int_{t_0}^{t_f} \{\, x^{*T}(t)Q(t)\tilde{x}(t) + u^{*T}(t)R(t)\tilde{u}(t)\,\}\, dt \;+\; x^{*T}(t_f)P_1\tilde{x}(t_f) \right]$$

$$+ \;\; \epsilon^2 \left[\int_{t_0}^{t_f} \{\, \tilde{x}^T(t)Q(t)\tilde{x}(t) + \tilde{u}^T(t)R(t)\tilde{u}(t)\,\}\, dt \;+\; \tilde{x}^T(t_f)P_1\tilde{x}(t_f) \right] \quad (4.12)$$

This expression holds for all possible $\tilde{u}(.)$ and all possible ϵ. Equation (4.12) has the form of

$$J = A + \epsilon B + \epsilon^2\, C$$

where A is the minimum of J and B and C are independent of ϵ and $C \geq 0$ (all scalars). As discussed in the homework problems, it follows that B, the coefficient of the ϵ term, must be zero (it is a necessary condition). That is

$$\int_{t_0}^{t_f} \{\, x^{*T}(t)Q(t)\tilde{x}(t) + u^{*T}(t)R(t)\tilde{u}(t)\,\}\, dt \;+\; x^{*T}(t_f)P_1\tilde{x}(t_f) = 0, \;\; \forall\, \tilde{u}. \;\; (4.13)$$

Now use (4.8) for $\tilde{x}$ in the above equation to obtain

$$\int_{t_0}^{t_f} \left[x^{*T}(t)Q(t) \int_{t_o}^{t} \Phi(t,\tau)B(\tau)\tilde{u}(\tau)d\tau \right] dt + \int_{t_o}^{t_f} u^{*T}(t)R(t)\tilde{u}(t)\, dt$$

$$+ \;\; x^{*T}(t_f)P_1 \int_{t_o}^{t_f} \Phi(t_f,t)B(t)\tilde{u}(t)dt = 0, \;\; \forall\, \tilde{u}.$$

We will concentrate on the first term. This term can be manipulated in the following form (through basic change of variables)

$$\int_{t_0}^{t_f} \int_{t_0}^{t} x^{*T}(t)Q(t)\Phi(t,\tau)B(\tau)\tilde{u}(\tau)d\tau\; dt \stackrel{\tau \leftrightarrow t}{=}$$

$$\int_{t_0}^{t_f} \int_{t_0}^{\tau} x^{*T}(\tau)Q(\tau)\Phi(\tau,t)B(t)\tilde{u}(t)dt\; d\tau$$

which, according to yet another homework problem (!), can be written as

$$\int_{t_0}^{t_f} \int_{t}^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\, B(t)\, \tilde{u}(t)\; d\tau\; dt.$$

Incorporating all in (4.13) we get the following necessary condition for $u^*(.)$ to be optimal

$$\int_{t_0}^{t_f} \left[\int_t^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\, B(t)\ d\tau + u^{*T}(t)\, R(t) \right] \tilde{u}(t)\ dt$$

$$+ \int_{t_0}^{t_f} \left[x^{*T}(t_f)\, P_1\, \Phi(t_f,t) B(t) \right] \tilde{u}(t)\ dt = 0. \tag{4.14}$$

Recall that equation (4.14) holds for all possible $\tilde{u}(.)$, which implies that the integrand must be zero, identically. (Technically, we should say almost everywhere!). Therefore, we have

$$\int_t^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\, B(t)\ d\tau + u^{*T}(t)\, R(t) + x^{*T}(t_f) P_1 \Phi(t_f,t) B(t) = 0. \tag{4.15}$$

Next, we will define the (so called co-state) vector $p(t)$ according to

$$p^T(t) \triangleq \int_t^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\ d\tau + x^{*T}(t_f) P_1 \Phi(t_f,t). \tag{4.16}$$

With this definition, (4.15) gives an expression for the optimal control law, since (4.15) and (4.16) imply

$$u^{*T}(t)\, R(t) + p^T(t) B(t) = 0 \Rightarrow u^{*T}(t) = -p^T(t) B(t) R^{-1}(t)$$

or

$$u^*(t) = -R^{-1}(t)\, B^T(t)\, p(t). \tag{4.17}$$

Note that we are still far from done. We need to find $p(t)$ and even then the control law appears to be open loop!! Now with the help of Leibniz rule (see homework problems!) we can find the derivative of $p(t)$ the so called co-state vector.

$$\begin{cases} \dot{p}(t) = -Q(t)\, x^*(t) - A^T(t)\, p(t) \\ p(t_f) = P_1 x^*(t_f). \end{cases} \tag{4.18}$$

Equation (4.18) is called the adjoint equation. Now using (4.17) in (4.4), we get the following set of differential equations

$$\begin{cases} \dot{x}^*(t) = A(t)\, x^*(t) - B(t)\, R^{-1}(t)\, B^T(t)\, p(t) \\ \dot{p}(t) = -Q(t)\, x^*(t) - A^T(t)\, p(t) \end{cases} \tag{4.19}$$

with the end condition

$$\begin{cases} x^*(t_o) = x_o \\ p(t_f) = P_1 x^*(t_f) \end{cases} \tag{4.20}$$

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Equations (4.19) and (4.20) constitute a set of 'two point boundary value' problem. A common way to approach this problem, i.e., the fact that $p(t_o)$ is not known, is to employ trial and error type techniques. In any rate, solving this set of equations can be formidable. Worse yet, it still leaves us with an open loop control!! These two problems force us to dig deeper!

We will go back to (4.19) and try to write the response! From (4.19) we have

$$\begin{pmatrix} x^*(t_f) \\ p(t_f) \end{pmatrix} = \Theta(t_f, t) \begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix}$$

where $\Theta(t_f, t)$ is the state transition matrix corresponding to the $2n$ order system of (4.19). Using the semi-group property of the state transition matrix, we can write

$$\begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix} = \Theta(t, t_f) \begin{pmatrix} x^*(t_f) \\ p(t_f) \end{pmatrix} = \begin{pmatrix} \theta_{11}(t, t_f) & \theta_{12}(t, t_f) \\ \theta_{21}(t, t_f) & \theta_{22}(t, t_f) \end{pmatrix} \begin{pmatrix} x^*(t_f) \\ p(t_f) \end{pmatrix}$$

where $\Theta(t, t_f)$ has been partitioned into 4 $n \times n$ blocks. Separating the two equations, and recalling that $p(t_f) = P_1 x^*(t_f)$, we have

$$x^*(t) = \theta_{11}(t, t_f)\, x^*(t_f) + \theta_{12}(t, t_f)\, P_1\, x^*(t_f) \tag{4.21}$$

$$p(t) = \theta_{21}(t, t_f)\, x^*(t_f) + \theta_{22}(t, t_f)\, P_1\, x^*(t_f) \tag{4.22}$$

Finding $x^*(t_f)$ from (4.21) and using it in (4.22), we get

$$p(t) = [\theta_{21}(t, t_f) + \theta_{22}(t, t_f)\, P_1]\, [\theta_{11}(t, t_f) + \theta_{12}(t, t_f)\, P_1]^{-1}\, x^*(t). \tag{4.23}$$

Note that the two brackets are independent of x_o and only depend on system matrices. This leads us to introduce

$$P(t) \triangleq [\theta_{21}(t, t_f) + \theta_{22}(t, t_f)\, P_1]\, [\theta_{11}(t, t_f) + \theta_{12}(t, t_f)\, P_1]^{-1}, \tag{4.24}$$

which results in the following form for the optimal control

$$p(t) = P(t)\, x^*(t) \Rightarrow u^*(t) = -R^{-1}(t)\, B^T(t)\, P(t)\, x^*(t) \tag{4.25}$$

which is in the feedback form!!! and is independent of x_o. The last problem is actually finding this $P(t)$! For this, we go back and differentiate (4.23) (i.e, $p(t) = P(t)x^*(t)$), which yields

$$\dot{p}(t) = \dot{P}(t)x^*(t) + P(t)\dot{x}^*(t)$$

using the expressions for $\dot{p}$ and $\dot{x}^*$, from (4.19),

$$-Q(t)x^*(t) - A^T(t)\, p(t) = \dot{P}(t)x^*(t) + P(t)A(t)x^*(t) - P(t)B(t)R^{-1}(t)p(t)$$

and using $p(t) = P(t)x^*(t)$, we get $\forall\, t \in [t_o, t_f]$

$$\left\{ \dot{P}(t) + P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t) \right\} x^*(t) = 0,$$

for all $x^*(.)$ that follow from all possible x_o's. Therefore, we must have $\forall\, t \in [t_o, t_f]$

$$\left\{ \begin{array}{l} \dot{P}(t) + P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t) = 0, \\ P(t_f) = P_1 \end{array} \right.$$
$$(4.26)$$

where the final condition is due to $p(t_f) = P_1 x^*(t_f)$. Equation (4.26) is known as the 'matrix (differential) Riccati' equation.

We can summarize the solution of LQR as follows:

ALGORITHM:
- solve (4.26) 'backwards in time' for $P(t)$
- store the gain matrix $K(t) = -R^{-1}(t)B^T(t)P(t)$
- implement $u(t) = K(t)x(t)$ on line

It appears that the problem is solved! (But we cannot just leave 'good enough' alone, can we?) We can come up with a lot of nifty results. For example, define

$$\left\{ \begin{array}{l} \mathcal{L}(t, t_f, x_o, u^*(.)) \triangleq x^{*T}(t)P(t)\,x^*(t) \\ \mathcal{L}(t_f, t_f, x_o, u^*(.)) = x^{*T}(t_f)P_1\,x^*(t_f) \end{array} \right. \qquad (4.27)$$

that is, a function that starts from t_f and evolved backward toward t_o, for a given optimal trajectory.

It can be shown (see homework problems) that the derivative of this function satisfies the following

$$\dot{\mathcal{L}}(t, t_f, x_o, u^*(.)) = -x^{*T}(t)Q(t)x^*(t) - u^{*T}(t)R(t)u^*(t)$$

which implies that $\mathcal{L}(t, t_f, x_o, u^*(.))$ is the value of J if only the portion from t to t_f was integrated. As a result, $J(t_o, t_f, x_o, u^*(.)) = \mathcal{L}(t_o, t_f, x_o, u^*(.))$ or

$$J_{min} = x^{*T}(t_o)P(t_o)x^*(t_o). \qquad (4.28)$$

This last result is quite useful for calculating the total 'cost' associated for a given set of initial conditions. It also leads to a great deal of insight into this optimal control problem.

We will now look into the case where $t \to \infty$. First, since from now on we are interested in very large terminal time, it makes sense to make the following assumption:

Assumption 4.2. *For indefinite horizon problem, $P_1 = 0$.*

Next consider two different values for the final time:

1) $t_f = t_2$

2) $t_f = t_1$

where $t_2 > t_1$. It can be shown (see HW) that

$$J(t_o, t_1, x_o) \leq J(t_o, t_2, x_o). \tag{4.29}$$

Note that we have eliminated u from the arguments of J since in the optimal case, $u(.)$ is a function of x. Now let us use the following definition:

$$\begin{cases} P^1(t) : The\ solution\ to\ (4.26)\ with\ t_f = t_1 \\ P^2(t) : The\ solution\ to\ (4.26)\ with\ t_f = t_2 \end{cases} \tag{4.30}$$

Clearly, (4.28), (4.29) and (4.30) imply that

$$P^1(t_o) \leq P^2(t_o) \cdots \quad .$$

This relations holds regardless of the value of x_o or behavior of the system matrices, $A(t)$, etc. Therefore we come to this 'bottom line':

Remark 4.3. *The longer you integrate the Riccati equation, the larger the value of $P(t_o)$.*

Now, we are ready to tackle the steady state case. While there are some applications for periodic systems, these results are mostly used for the time-invariant case. The steady state case is sometimes called 'the infinite horizon' case.

4.2   The Steady State Riccati Equation (SSRE)

Since we will deal with time invariant systems (from now on, that is), we can use $t_o = 0$ with out loss of generality. Also, note that from now on, we will use the following assumption:

Assumption 4.4. *Matrices A,B, Q, and R are all constant matrices.*

The problem, therefore is the following:

$$\begin{cases} \dot{x}(t) = Ax + Bu \\ x(0) = x_o \end{cases} \qquad (4.31)$$

where we have dropped (t) from x and u, for simplicity. The cost function to be minimized is

$$J(x_o) = \int_0^\infty \{\, x^T Q x + u^T R u \,\} \, dt. \qquad (4.32)$$

From the development of previous Section, we know that the optimal solution has the following form for the control law

$$u_{opt}(t) = -R^{-1} B^T P(t) x(t) \qquad (4.33)$$

where $P(t)$ is obtained from

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q = 0, \quad \forall\, t \in [t, \infty] \\ P(\infty) = 0. \end{cases}$$
$$(4.34)$$

By the discussion at the end of the previous Subsection, it is clear that as (4.34) is integrated backwards

$$P(t_2) \le P(t_1) \quad if \quad t_2 \ge t_1$$

(because the duration of integration for $P(t_1)$ is longer). The question becomes: "As we integrate more and more, does P (a nondecreasing function) blow up or does it converge to something?" Note that by (4.28), $P(0)$ have to be at least semi-positive definite.

Next, suppose that (A, B) is stabilizable; i.e., $\exists$ a gain matrix such that $(A - BK)$ is stable. Now use this gain for control; i.e., use $u = -Kx$, which results in a closed loop system of $\dot{x} = (A - BK)x$. Clearly, this may not be the optimal control. Let us calculate the cost functional J for this control law

$$J = \int_0^\infty \{\, x^T Q x + u^T K^T R K x \,\} \, dt = \int_0^\infty \{\, x^T (Q + K^T R K) x \,\} \, dt.$$

However, for this system we know that $x(t)$ has the from

$$x(t) = e^{(A-BK)t} x_o$$

which results in

$$J = \int_0^\infty \{ x_o^T e^{(A-BK)^T t} (Q + K^T R K) e^{(A-BK)t} x_o \} \, dt.$$

From basic definitions of stability and exponential stability, it follows that this J is finite. Since this is not necessarily the optimal J, the optimal J is finite, as well. By (4.28), it follows that P does not blow up! We make the following observation:

Remark 4.5. *Stabilizability implies the existence of an optimal control law. Indeed, it also produces upper bounds for J and, hence, $P(0)$. Therefore, it also implies the convergence of $P(t)$.*

Once this convergence occurs, say $P(t) \to P$, the derivative in (4.34) disappears and P satisfies the following equation (known as the 'Algebraic Riccati Equation or ARE')

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \quad ARE \tag{4.35}$$

The convergence can be shown "formally", by assuming $P(t)$ converges to some $\overline{P}$ (due to convergence of J). Then it can be shown that this is the same as P in (4.35), for any positive semi definite solution of (4.35).

Remark 4.6. *Uniqueness of the positive semi-definite solution to (4.35) - and stability of the closed loop sysem - is established through the use of another assumption: (A, Q) observable.*

4.3 The Potter's Method

Let us start with the optimal case

$$\dot{x}(t) = Ax + Bu = \left(A - BR^{-1}B^T P\right) x$$

and for brevity, define

$$G \stackrel{\triangle}{=} BR^{-1}B^T \tag{4.36}$$

so that the closed loop system and ARE can re written as

$$\begin{cases} \dot{x}(t) = (A - GP)x \\ PA + A^T P - PGP + Q = 0. \end{cases} \tag{4.37}$$

Matrix $(A - GP)$ is the closed loop matrix. Its eigenvalues and eigenvectors are called the closed loop eigenvalues and eigenvector, respectively. Denote the eigenvalues and eigenvectors of the closed loop system by

$$(A - GP)X_i = \lambda_i X_i. \tag{4.38}$$

From (4.37), we have $P(A - GP) = -A^T P - Q$, therefore (using (4.38))

$$P(A - GP)X_i = \lambda_i P X_i = -A^T P X_i - Q X_i. \tag{4.39}$$

Now, putting (4.38) and (4.39) in matrix form

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \left\{ \begin{array}{c} X_i \\ P X_i \end{array} \right\} = \lambda_i \left\{ \begin{array}{c} X_i \\ P X_i \end{array} \right\}. \tag{4.40}$$

The $2x \times 2x$ matrix in (4.40) is called the 'Hamiltonian' matrix. From now on, we will call it H ; i.e.,

$$H \stackrel{\triangle}{=} \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}. \tag{4.41}$$

From (4.40), we see that every closed loop eigenvalue is an eigenvalue of H. Also, note the relationship between the top and bottom halves of the eigenvectors of H corresponding to these eigenvalues. If everything goes o.k., we will be looking for stable eigenvalues of H (better have stable closed loop!) and exploit the structure of these eigenvectors. Note that if we call the bottom of each stable eigenvector $Y_i = PX_i$, then by stacking these n vector equations next to one another, we have

$$[\, Y_1 \ Y_2 \ \ldots Y_n \,] = P \, [\, X_1 \ X_2 \ \ldots X_n \,].$$

That is how Potter solved the problem about 30 years ago. Consider this algorithm:

- Form H in (4.41) (assume $Re(\lambda(H)) \neq 0$)

- Solve for eigenvalues and eigenvectors. Choose the stable ones only.

- Stack the stable eigenvectors next to one another to form a $2x \times n$ matrix. Call the top half X and the bottom half Y.

- Find P according to $P = Y\,X^{-1}$

It is easy to show (see HW, also it is one of the main properties of Hamiltonian type matrices) that if λ is an eigenvalue of H, so is $-\lambda$ (i.e., eigenvalues are symmetric with respect to the imaginary axis). That is, there cannot be more that n stable (or unstable) eigenvalues for H!

What are the pitfalls? What if some of the eigenvalues of H have zero real parts? What if X is not invertible? We will deal with these through homework problems and class discussion.

Remark 4.7. *There has been a great deal of research on the minimum (i.e., sufficient and necessary) conditions needed for the existence of a $P \geq 0$ that stabilizes the closed loop (ans: detectability and stabilizability), those needed to have $P > 0$, efficient numerical algorithms (e.g., Schur methods), etc. In class discussions, we will address some of these issues and talk about references for most others.*

4.4 PROBLEM SET

P1. When we write $x^T Q x$, or similar expressions, we typically assume Q is symmetric. Show that this is not an important -or restrictive - assumption, as long as Q is used in these quadratic forms only.

P2. Let $X(\epsilon) = A + \epsilon B + \epsilon^2 C$, with $A > 0$, $C \geq 0$ and ϵ scalar (possibly negative). Also, A, B, C are independent of ϵ. Show that if A is the minimum value of $X(\epsilon)$, for all ϵ, then B must be zero.

P3. Show that

$$\int_{t_o}^{t_1} \int_{t_o}^{\tau} A(t,\tau) \, dt \, d\tau = \int_{t_o}^{t_1} \int_{t}^{t_1} A(t,\tau) \, d\tau \, dt$$

P4. Let $\Phi(t,\tau)$ be the state transition matrix for $A(t)$. Show that

$$\frac{d}{dt}\Phi(\tau,t) = -\Phi(\tau,t)A(t)$$

P5. If $p(t)$ is defined by (4.16), use Leibniz rule to show that (4.18) is true.

P6. Show that in (4.26), the solution P is symmetric; i.e., $P(t) = P^T(t)$

P7. calculate and simplify the derivative of $\mathcal{L}$ defined in (4.27). Show (4.28) holds (hint: consider $\hat{J} = \int_t^{t_f} x^T Q x \cdots$)

P8. Show that (4.29) holds (remember to set P_1 to zero).

P9. Show that the eigenvalues of H in (4.41) are symmetric with respect to the imaginary axis. (hint: one way is to lok into the possibility of having $(-X^T P \quad X^T)^T$ as eigenvector of H^T)

P10. Show that if (A, B) is controllable and (A, Q) is observable, then eigenvalues of H will not have zero real parts.

P11. Using (4.35), and not the Hamiltonian, show that

$$Q > 0 \Rightarrow P > 0 \quad and/or \quad (A, Q) \ observable \quad \Rightarrow P > 0$$

(think of J). What about

$$Q > 0 \Rightarrow (A - GP) \ stable$$

$$Q \geq 0, \quad (A, Q) \ observable \Rightarrow (A - GP) \ stable$$

Can you use the basic definition of J to interpret these last results?