5 EXTENSIONS TO LQR

We start with the *infinite horizon, time invariant* problem, which has been discussed in some details. One potential problem may be that while the LQR problem we discussed minimizes the cost functional, it may not result in good response (e.g., the state vector dies too slowly) or does not fit our goals (our objectives do not fit the form of J we have been using). In this Section, we will try to deal with a few of these issues. Unless specifically needed, we will simplify notations by using x, y, etc. instead of $x(t)$, $y(t)$, etc. The time dependency will be clear from the context.

5.1 Cross Terms in the Cost Functional

Consider the same system as before, i.e,

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_o. \end{cases} \tag{5.1}$$

Instead of the J we used before, however, let us try to minimize the following cost functional

$$J(x_o) = \int_0^\infty \{\, x^T Qx + u^T Ru + 2x^T Su \,\}\, dt. \tag{5.2}$$

where S is a matrix of appropriate dimension.

Remark 5.1. *Cost functionals of the form* (5.2) *are encountered when, for example, one is interested in minimizing the integral of control effort (i.e., the $u^T Ru$ term) plus the square of the norm of some outputs $y = Cx + Du$. The term $\|y\|^2$ would, typically, have cross terms.*

The first step in solving this problem is to do the following manipulations (of the 'completing the square' variety);

$$u^T Ru + 2x^T Su + x^T Qx = (u + R^{-1}S^T x)^T R\, (u + R^{-1}S^T x) + x^T (Q - SR^{-1}S^T)x. \tag{5.3}$$

Next, define the following modified dynamics

$$\begin{cases} \dot{x} = (A - BR^{-1}S^T)x + B\tilde{u} \\ \tilde{u} = u + R^{-1}S^T x. \end{cases} \tag{5.4}$$

The cost funcctional in (5.2) can now be written as

$$J(x_o) = \int_0^\infty \{\, x^T (Q - SR^{-1}S^T)x + \tilde{u}^T R\tilde{u} \,\}\, dt. \tag{5.5}$$

Note that (5.4) and (5.5) form a standard LQR problem, i.e., one obtains the positive definite solution of

$$P(A - BR^{-1}S^T) + (A - BR^{-1}S^T)^T P - PBR^{-1}B^T P + Q - SR^{-1}S^T = 0$$

and implements the control $\tilde{u} = -R^{-1}B^T Px$ in (5.4) to minimizes (5.2) or (5.5). Implementing this control law in (5.4), however, is equivalent to using the following control law in our original sysem (i.e., (5.1))

$$u(t) = \tilde{u}(t) - R^{-1}S^T x(t) = -R^{-1}(B^T P + S^T)x(t). \tag{5.6}$$

Finally, for the problem to work, we need to make the following assumptions:

Assumption 5.2. *Matrix S is chosen (e.g., small enough) so that $\tilde{Q} = Q - SR^{-1}S^T \geq 0$.*

Assumption 5.3. *The pair $[(A - BR^{-1}S^T)\ ,\ \tilde{Q}]$, is observable and the pair $[A, B]$ is controllable.*

Remark 5.4. *The first assumption above is needed for the cost functional* (5.5) *to make sense. The second one is needed for technical (yet important) reasons, such as stability of the closed loop. Both can be met by having S small enough, if $Q > 0$. Also note that we do not need to require $[(A - BR^{-1}S^T), B]$ to be controllable, if $[A, B]$ was controllable to start with.*

5.2 Regulators with a Prescribed Degree of Stability - α shifts

Recall that in the infinite horizon problem, we traded the ability to set the terminal time with the ability to solve the ARE and, hence, come up a much easier controller (both to calculate and to implement). However, one may ask: what if the resulting problem reduced x at unacceptably slow rates? In this subsection, we deal with the case where we wish to kill the state faster than the standard LQR. To do this, we modify the cost functional. Again we start with the dynamics

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax + Bu \\ x(0) = x_o \end{array} \right. \tag{5.7}$$

but try to minimize

$$J(x_o) = \int_0^\infty \{ \, e^{2\alpha t}(x^T Q x + u^T R u) \, \} \, dt, \qquad \alpha \geq 0. \tag{5.8}$$

Note that this cost functional will try to force x to die at least as fast as $e^{-\alpha t}$ (why?) The constant α, therefore, can be used to force the controller to act faster! The problem is that (5.8) creates a time varying LQR problem, which destroys a great deal of convenience. The question is: Can we trick the controller so that is solves (5.7) and (5.8) by trying a related time invariant LQR problem? The answer, as you may have guessed, is yes!

First, we need to define

$$\left\{ \begin{array}{l} \hat{x}(t) = e^{\alpha t} \; x(t) \\ \hat{u}(t) = e^{\alpha t} \; u(t). \end{array} \right. \tag{5.9}$$

With these definitions, the cost functional in (5.8) can be written as

$$J(x_o) = \int_0^\infty \{ \, (\hat{x}^T Q \hat{x} + \hat{u}^T R \hat{u}) \, \} \, dt \stackrel{\triangle}{=} \hat{J}(x_o). \tag{5.10}$$

Now, taking the derivative of $\hat{x}$ in (5.9), we obtain

$$\dot{\hat{x}}(t) = \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t) = \alpha \hat{x}(t) + A\hat{x}(t) + B\hat{u}(t)$$

or

$$\left\{ \begin{array}{l} \dot{\hat{x}}(t) = [A + \alpha I]\hat{x}(t) + B\hat{u}(t) = \hat{A}\hat{x}(t) + B\hat{u}(t) \\ \hat{x}(0) = x_o \end{array} \right. \tag{5.11}$$

Note that (5.11) and (5.7) are equivalent; i.e, one implies another, as long as we used the definitions in (5.9). Indeed, it is an easy exercise to show that a given $\hat{u}$ in (5.11) results in a $\hat{x}$ which is exactly $e^{\alpha t}$ times the x that results from implementing $u = e^{-\alpha t}\hat{u}$ in (5.7)!

As a result, our original problem is tranformed into minimizing (5.10), subject to (5.11), which looks like a standard time invariant LQR problem. Therefore, we solve for the positive definite solution of

$$P_\alpha(A+\alpha I)+(A+\alpha I)^T P_\alpha - P_\alpha B R^{-1} B^T P_\alpha + Q = 0 \tag{5.12}$$

where the subscript α is used to underline the fact that P depends on α. Next, we can write the the optimal control law

$$\hat{u}(t) = -R^{-1} B^T P_\alpha \hat{x}(t). \tag{5.13}$$

The last step is to find the control law for the actual system (i.e., (5.7)). For this, simply note that

$$u(t) = e^{-\alpha t}\hat{u}(t) = -e^{\alpha t} R^{-1} B^T P_\alpha \hat{x}(t) = -R^{-1} B^T P_\alpha x(t). \tag{5.14}$$

Remark 5.5. *The only change from the standard case is that the ARE has been changed (hence, P_α). Since it is easy to show that the controllability and observability are not affected by replacing A with $(A+\alpha I)$, controllability and observability of the original system implies the same properties for the system in* (5.10) *and* (5.11)*. Note, however, that this is not necessarily true for stabilizability and detectability.*

Remark 5.6. *This method is often called 'the α shift' for obvious reasons. You can say that by using $A+\alpha I$, we are pretending our system to be a lot more unstable (or less stable). As a result, the control will 'work harder' to push everything further back to the left half plane, since it tries to find a K such that the eigenvalues of $A+BK+\alpha I$ are in the left half plane. This imples that the same K would result in closed loop (i.e., $A+BK$) eigenvalues with real parts less than $-\alpha$.*

5.3 The Servo and Tracking Problems

Consider the follwoing system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5.15}$$

where y is the measured output of the system. The servo and tracking problems concern the issue of following a trajectory $\tilde{y}(t)$, by minimizing the following cost functional

$$J(x_o) = \int_0^T \{\, (y - \tilde{y})^T Q (y - \tilde{y}) + u^T R u \,\}\, dt. \tag{5.16}$$

where the composition of J is motivated by the need to reduce error between desired trajectory and output, while having some flexibility with respect to control effort. We will use the terminal time T so that finite duration control can also be attempted.

The desired trajectory, $\tilde{y}$, may be constant or time varying. The treatment here deals with both. For the sake of simplicity, however, we have dropped the explicit dependence on t. A typical approach is to first generalize this cost functional into

$$J(x_o) = \int_0^T \{\, (y - \tilde{y})^T Q_2 (y - \tilde{y}) + \bar{y}^T Q_1 \bar{y} + u^T R u \,\}\, dt, \quad Q \geq 0, Q_1 \geq 0, \tag{5.17}$$

where

$$\begin{cases} \bar{y} = \bar{C}x \\ \bar{C} = I - LC = I - C^T (CC^T)^{-1}\, C. \end{cases} \tag{5.18}$$

Remark 5.7. *The generalization in* (5.17) *can be made without any loss of generality since we can always set* $Q_1 = 0$*.*

Based on the definitions in (5.18), we have

$$\begin{cases} CL = I, \quad\quad C\bar{y} = 0 \\ x = \bar{y} + x_1 \quad\quad , \quad x_1 \in Range(C^T). \end{cases} \tag{5.19}$$

Remark 5.8. *Considering the basic definitions of orthogonal projections, it is simple to see that* $\bar{C}$ *is the (orthogonal) projection matrix onto the othogonal complement of range space of* C^T*; i.e, the null space of* C*. In other word,* $\bar{y}$ *is part of the state vector that is not seen by* $y = Cx$*.*

Next, we define

$$\begin{cases} \tilde{x} = L\tilde{y} \\ which \;\Rightarrow \tilde{y} = C\tilde{x}. \end{cases} \tag{5.20}$$

Note that by (5.20), we have 'found' a state trajectory that results in our desired output trajectory, if passed through C. As a result, we can turn the output error based cost functional of (5.16) or (5.17) with the following, state tracking error based, cost functional

$$\begin{aligned} J(x_o) &= \int_0^T \{ (x-\tilde{x})^T [C^T Q_2 C + \bar{C}^T Q_1 \bar{C}](x-\tilde{x}) + u^T R u \} \, dt \\ &= \int_0^T \{ (x-\tilde{x})^T Q (x-\tilde{x}) + u^T R u \} \, dt. \end{aligned} \tag{5.21}$$

As a result, from now on, we can (without any loss of generality) focus on problems that have state trajectory error terms in their cost functionals.

5.3.1 The Servo Problem

Let us assume that trajectory is (or could be!) from the following model

$$\begin{cases} \dot{z} = Fz \\ \tilde{y} = Hz \ , \quad (F, H) \ \ \textit{observable} \end{cases} \tag{5.22}$$

for some H and F. Note that z plays the same role as $\tilde{x}$ in (5.20). The objective here is to minimize costs of the form (5.21), with $\tilde{x}, L, etc.$ as discussed previously.

Remark 5.9. *Equation* (5.22) *can be used to model a great variety of output trajectories. For example: every polynomial function of time (of any order).*

We can distinguish two cases:

Case 1 z is directly available:

The original system and the model in (5.22) - as well as the cost functional - can be written in terms of the following augmented system

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{J} = \int_0^T \{ \hat{x}^T \hat{Q} \hat{x} + u^T R u \} \, dt, \end{cases} \tag{5.23}$$

where we have used

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} \ , \ \hat{B} = \begin{pmatrix} B \\ 0 \end{pmatrix} \ , \ \hat{Q} = \begin{pmatrix} Q & -QLH \\ -H^T L^T Q & H^T L^T Q L H \end{pmatrix}. \tag{5.24}$$

From the basic LQR, we know the optimal control law is

$$u^*(t) = -R^{-1} \hat{B}^T \hat{P}(t) \ \hat{x}(t) \tag{5.25}$$

where $\hat{P}(t)$ is the solution to the Riccati matrix differential equation

$$\begin{cases} -\dot{\hat{P}}(t) = \hat{A}^T\hat{P}(t) + \hat{P}(t)\hat{A} - \hat{P}(t)\hat{B}^T R^{-1}\hat{B}\hat{P}(t) + \hat{Q} \\ \hat{P}(T) = 0. \end{cases} \tag{5.26}$$

To simplify the controller, let us partition $\hat{P}(t)$

$$\hat{P}(t) = \begin{bmatrix} P(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix}. \tag{5.27}$$

We can write the control law of (5.25) as

$$\begin{cases} u^*(t) = K_1(t)x(t) + K_2(t)z(t) \\ K_1(t) = -R^{-1}B^T P(t) \\ K_2(t) = -R^{-1}B^T P_{12}(t) \end{cases} \tag{5.28}$$

with

$$\begin{cases} -\dot{P}(t) = P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q \\ -\dot{P}_{12}(t) = P_{12}(t)F + A^T P_{12}(t) - P(t)BR^{-1}B^T P_{12}(t) - QLH \\ -\dot{P}_{22}(t) = P_{22}(t)F + F^T P_{22}(t) - P_{12}(t)BR^{-1}B^T P_{12}(t) + H^T L^T QLH \\ P(T) = P_{12}(T) = P_{22}(T) = 0. \end{cases} \tag{5.29}$$

Similarly, it can be shown that the final cost wil be

$$J^* = x_o^T P(0)x_o + 2x_o^T P_{12}(0)z(0) + z^T(0)P_{22}(0)z(0). \tag{5.30}$$

Remark 5.10. *The initial time of zero was used for simplicity. If needed, simply replace the appropriate t_o in the usual places (e.g., in the cost functionals and in* (5.30)*). Similarly, we have used no terminal time penalty. This can be added easily. Start with a P_1 to penalize the error in the state tracking at time T, find the corresponding $\hat{P}_1$ for* (5.23)-(5.25)*. It will only affect, directly, the final conditions in* (5.29)*.*

Remark 5.11. *Since the duration is finite, the control is time varying. However, by inspecting* (5.28)*, you can see that the control has two parts. One is the exact same term you would get out of standard LQR (i.e., if you were interested in killing the state and not tracking). The secon part was only dependent on P_{12} and z. In effect, this is a feedforward type of compensation! Lastly, note that the equation for P_{22} need not be integrated for control, since it is not used in either of the other two equations. It is only used to evaluate the total cost in* (5.30)*.*

Case 2 z is not available directly:

We simply desgin an observer for the model (5.22) !! The rest is identical (but much messier!)

5.3.2 The Tracking Problem

In the last subsection, we discussed the servo problem, in which we assumed the existence of a model (or another system) which is the sourse of the desired trajectory (i.e., (5.22)). As a result, we require the knowledge of the model (i.e., F and H) and on-line measurement of z (the desired state trajectory). Now, let us consider the case where the knowledge of F and H is not possible, but we may know the complete trajectory history (i.e., we know $\tilde{y}(t)$ for all $t \in [0, T]$).

By remark 5.11, we know that the optimal control has two parts. The part that we will have problems with is the feedforward term, since its evaluation required $P_{12}(t)$ which itself needed H and F, see (5.29). So we will start with the feedformard term

$$u_{ff}(t) = -R^{-1}B^T P_{12}(t)z(t) = -R^{-1}B^T b(t)$$

where we have defined $b = P_{12}z$. Let us take derivative of this vector. After some minor manipulations, we have

$$-\dot{b}(t) = (A - BR^{-1}B^T P(t))^T b(t) - QL\tilde{y}(t) \quad , \; b(T) = P_{12}(T)z(T) = 0, \tag{5.31}$$

where equations (5.22) and (5.29) are used.

The tracking problem is then solved by integrating (5.31) backward in time from T to zero and implementing the follwoing control law

$$u^*(t) = -R^{-1}B^T[P(t)x(t) + b(t)].$$

Remark 5.12. *If we let $T \to \infty$, certain simplifications can be made. For example, the backward integration in* (5.31) *will be well behaved (since $(A - BR^{-1}B^T P)$ is the closed loop LQR matrix and stable). Also, if $\tilde{y}$ is constant, then $b(t)$ actually converges to a constant (think of the integration in* (5.31) *as the response of a stable system to constant input) and the feedforward term becomes a constant, as well (an offset!).*

5.4 PROBLEM SET

Exercise 5.13. *Show that if* (A, B) *is controllable, so is* $[(A - BR^{-1}S^T), B]$.

Exercise 5.14. *Show the details of the tranformation of* (5.2) *to* (5.5).

Exercise 5.15. *Show that controllability and observability are not affected by replacing* A *with* $A + \alpha I$. *What about detectability and stabilizability?*

Exercise 5.16. *What is the rate of decay for* $x(t)$ *if* (5.14) *is used? Show it by (1) using* $\hat{x}$ *and (2) Lyapunov arguments based on behavior of* $x^T P_\alpha x$ *where* P_α *is the solution of* (5.12).

Exercise 5.17. *What is the value of* J_α *in terms of* P_α*? Is* $J_\alpha \geq J_{\alpha=0}$ *if* $\alpha \geq 0$*? Hint: Take* $\frac{d}{d\alpha}$ *of* (5.12) *and show* $\frac{dP_\alpha}{d\alpha} > 0$ *through a Lyapunov argument. Then consider* $\frac{d}{d\alpha} J_\alpha$.

Exercise 5.18. *Consider* (5.18). *Show that* LC *is the orthogonal projection operator onto the range of* C^T *and, hence,* $I - LC$ *is the projection onto the null spave of* C.

Exercise 5.19. *Again, in* (5.18), *show* $(LC)(LC) = LC$, $(I - LC)LC = 0$.

Exercise 5.20. *Show* (5.17) *and* (5.21) *are the same.*

Exercise 5.21. *Verify* remark 5.9. *Hint: Consider an* F *with all entries zero with ones on the super diagonal.*

Exercise 5.22. *Verify* (5.23),(5.24) *and* (5.28)-(5.30).

Exercise 5.23. *What happens if there is a terminal penalty term for the servo problem. Work the details. How does it effect the tracking problem.*

Exercise 5.24. *Work the details of the* remark 5.12.