6 OUTPUT FEEDBACK DESIGN

When the whole sate vector is not available for feedback, i.e, we can measure only

$$y = Cx.$$

6.1 Review of observer design

Recall from the first class in linear systems that a simple control law would be

$$u = Kx \Longrightarrow \dot{x} = (A + BK)x$$

where K is chosen so that $A+BK$ is stable (from pole placement of LQR, etc). Now if you can not measure x, then you use an output feedback design. Static output feedback design; i.e,. $u = Ky$ turns out to be relatively hard to solve (unless you do trial and error) - more on this later. The most common - and systematic approach is to use a dynamic output feedback, where the controller (or compensator) has its own dynamics (recall the typical compensator box from classical control course). The simplest form is an observer structure; i.e., use $u = K\hat{x}$ where $\hat{x}$ is an estimate for the actual x and comes from a copy of the model we construct with our control hardware (or software)

$$\dot{x} = Ax + Bu \tag{6.1}$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - C\hat{x}) \tag{6.2}$$

$$u = K\hat{x} \tag{6.3}$$

The trick, in this simple approach, is to pick a good L such that $\hat{x} \longrightarrow x$ relatively soon. First, let us write the model in terms of x and $e = x - \hat{x}$

$$\dot{x} = Ax + BK\hat{x} = (A + BK)x - BKe \tag{6.4}$$

$$\dot{e} = (A + LC)e \tag{6.5}$$

$$\tag{6.6}$$

which can be written as the following for $x_{cl}^T = (x^T \quad e^T)$

$$\dot{x}_{cl} = A_c x_{cl} \quad , \quad A_{cl} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix}$$

since $del(A_{cl}) = det(A + BK) \; . \; det(A + LC)$, we have closed loop stability (i.e. $e(t) \longrightarrow 0$ as time gets larger, which means $\hat{x} \longrightarrow x$) as long as $A + LC$ stable. Note that this is independent of the choice of K - a trivial case of 'separation principle,' Also note that the poles of $A + LC$ set how fast $e(t)$ dies. It is common to use a role of thumb that days the least stable pole of $A + LC$ should be three times as fast as the dominant modes of $A + BK$.

6.2 The Kalman filter and the LQG

In this subsection, we will review -very briefly- Kalman filter equations for the Linear Quadratic Guassian problem (LQG). Due to the time limitation, this review will be extremely brief.

Consider the following stochastic linear system

$$\begin{cases} \dot{x} = Ax + Bu + \Gamma\zeta(t) \\ y = Cx + \eta(t) \end{cases} \tag{6.7}$$

where $\zeta(t)$ and $\eta(t)$ are vector random processes (i.e., the process noise and measurement noise, respectively).

After ignoring a great deal of effort (and potential pitfalls) with respect to the well-posedness of (6.7), we assign

$$\begin{cases} \mathcal{E}\left[\zeta(t)\right] = \mathcal{E}\left[\eta(t)\right] = 0 \quad (\textit{zero mean}) \\ \mathcal{E}\left[\zeta(t)\eta^T(\tau)\right] = 0 \quad (\textit{uncorrelated}) \\ \mathcal{E}\left[\zeta(t)\zeta^T(\tau)\right] = Q_o \ \delta(t-\tau)\ , \ Q_o \geq 0 \\ \mathcal{E}\left[\eta(t)\eta^T(\tau)\right] = R_o \ \delta(t-\tau)\ , \ R_o > 0 \end{cases} \tag{6.8}$$

where $\mathcal{E}$ is the expectation operator (think ensemble avarage, or in the case of ergotic signals, time averages). Relying only on the measurement $y(t)$, we wish to 'estimate' the $x(t)$ - and denote the estimate by $\hat{x}$ - such that the following error is minimized:

$$\begin{cases} e(t) = x(t) - \hat{x}(t) \\ \textit{minimize} \quad \mathcal{E}\left[e^T(t)e(t)\right] = \textit{min}\ \mathcal{E}\left[\|x(t) - \hat{x}(t)\|^2\right]. \end{cases} \tag{6.9}$$

After a great deal of work (roughly two quarters of stochastic processes worth!) the following is obtained for the *steady state* case:

<u>Observer Equation</u> :

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f[y(t) - C\hat{x}(t)] \tag{6.10}$$

$$K_f = SC^T R_o^{-1} \tag{6.11}$$

$$AS + SA^T - SC^T R_o^{-1} CS + \Gamma Q_o \Gamma^T = 0. \tag{6.12}$$

Note that with this obsever, the error equation becomes

$$\dot{e}(t) = \left[A - K_f C\right] e(t) + \left[\Gamma \quad -K_f\right] \begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix}. \tag{6.13}$$

After noting the similarity of these equations to those of the LQR method, we have the following :

Remark 6.1. *The Riccati equation in* (6.12) *resembles the one encountered in LQR. Notice the duality between the two, by replacing B with C^T and A by A^T. As a result, a great many of the results and techniques we discussed earlier apply here, as well. For example: if (A, C) is observable and (A, Γ) is controllable, then* (6.12) *has a unique solution, $S > 0$, and $(A - K_f C)$ is stable.*

Remark 6.2. *In* (6.10) *if the noise terms are ignored, then we have an observer which is obtained from* (6.11)-(6.12)*, and results in a stable closed loop. Further, if the model has noise (as in* (6.7)*) then this observer minimizes* (6.9)*, as well. This is the approach we will choose; i.e., we will use* (6.12) *and* (6.11) *to desgin 'desirable' observers (rather than pole placement methods, for example). Lastly, matrices Q_o and R_o can be interpreted as the intensity of the process noise and measurement noise, respectively. A very large R_o, for example, denotes high levels of measurement noise. One might expect that such a system would end up with small observer gains (why?), which is indeed true. What is the dual problem in LQR?*

6.3 The Linear Quadratic Guassian Compensator - LQG

Consider the dynamical system in (6.7), subject to (6.8). The LQG problem is to design a controller of the form

$$u(t) = f[\ y(\tau)\ ,\ \tau \leq t\ ] \tag{6.14}$$

to minimize

$$J = \mathcal{E}[\lim_{T\to\infty} \frac{1}{T} \int_o^T (\, x^T H^T H x + u^T R u \,)\, dt] \tag{6.15}$$

under the following assumptions

$$\begin{cases} (A, B), \ \ and \ \ (A\Gamma)\ controllable(stabilizable) \\ (A, C) \ \ and \ \ (A, H)\ observable \\ R_o > 0 \ \ , \ \ \ R > 0 \ \ Q_o \geq 0. \end{cases} \tag{6.16}$$

The solution to this problem is the following:

$$\begin{cases} u(t) = -K_c \hat{x}(t) \\ K_c = R^{-1} B^T P \\ PA + A^T P - PBR^{-1}B^T P + H^T H = 0 \\ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f[y(t) - C\hat{x}(t)] \\ K_f = SC^T R_o^{-1} \\ AS + SA^T - SC^T R_o^{-1} CS + \Gamma Q_o \Gamma^T = 0. \end{cases} \tag{6.17}$$

Remark 6.3. *Equation* (6.17) *imples that the optimal solution can be separated into full state controller and observer design. This principle of separation in stochastic control works similar to the one encountered in pole-placement type controllers (in deterministic setting). Its proof, however, is quite complicated. Note the control consists of a LQR step plus an observer step (which is the steady state Kalman filter).*

6.4 Problem Set

Exercise 6.4. *Ignoring the noise (i.e., η and ζ), write the combined closed-loop state space form (in terms of state varaibles x and $\hat{x}$).*

Exercise 6.5. *Define the error to be $x - \hat{x}$. Write the combined closed-loop state state form in terms of state variables x and e. Are the eigenvalues of the closed loop system the same as in the previous exercise? Why?*

Exercise 6.6. *Again, set the noise to be zero. What is the transfer function of the compensator? That is, if we write the control as in $u(s) = H(s)\, y(s)$, what is $H(s)$, where '$x(s)$' denotes the Laplace Transform of $x(t)$. Try to draw the block diagram of this problem.*

Exercise 6.7. *Is the compensator (i.e., $H(s)$) stable? What are some of the possible problems with unstable compensators?*