7 MIMO ZEROS

We are interested in MIMO zeros for several reasons. Similar to the SISO case, nonminimum-phase zeros will effect the performance (degradation) of the control system. As we will see, the asymptotic behavior of the Riccati equation depends on the place of the zeros of certain transfer functions. Such results are somewhat similar to the SISO root-locus arguments on the poles as the gain becomes large (and the poles go to the open loop zeros).

Nonminimum phase zeros are, unfortunately, common in real life. A common example is a flexible beam with non-colocated actuator and sensor.

7.1 Transfer Function Approach

We start by going back to the single-input single-output (SISO) case. Consider the transfer function

$$G(s) = \frac{s+2}{(s+3)(s+4)}.$$

Now choose the input to be of the form $u(t) = u_o\, e^{-2t}$. It is simple to show that this input results in an output of the form $y(t) = u_o e^{-3t} - u_o e^{-4t}$; i.e., the output does not contain any terms of the form e^{-2t}. If the input had a term e^{-6t}, for example, the output *would* have the term e^{-6t} in it. Indeed, this can be generalized to second order terms (complex roots). Due to this property, we can think of the above development as if the system had 'blocked' the transmission of the signal of the form e^{-2t}.

This approach can be generalized to multi-input multi-outputs (MIMO) system by using the following for the input

$$u(t) = \underline{u}_k\, e^{z_k t}$$

where $\underline{u}_k$ and z_k are possibly complex vector and scalar, respectively. Now the question becomes: Are there any *pairs* $(\underline{u}_k, z_k)$ such that the output of a MIMO plant does not contain a $e^{z_k t}$ term?

Example 7.1. *Consider the plant*

$$y(s) = G(s)u(s) = \begin{bmatrix} \frac{s+1}{s^2} & 0 \\ 0 & \frac{s+2}{s^2} \end{bmatrix} u(s)$$

where it appears that $s = -1$ and $s = -2$ are zeros of the system. Indeed, it is easy to show that

$$u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \Rightarrow y(t) \text{ without an } e^{-t} \text{ term}$$

and

$$u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \Rightarrow y(t) \text{ without an } e^{-2t} \text{ term.}$$

Definition 7.2. *A system, $G(s)$, has a transmission zero at $s = z_k$ if there exist vector u_k and scalar z_k such that $u(t) = u_k\, e^{z_k t}$ results in the output $y(t)$ not containing $e^{z_k t}$ terms. The vector u_k is called the (zero) direction of z_k.*

Using slightly different definitions, different kinds of zeros have been introduced, such as 'blocking zero' or 'transmission zero' or 'transmission blocking zeros' etc. If $G(s)$ is minimal (i.e., controllable and observable) then there is only one kind of zero, whatever it is called.

Definition 7.3. *A unimodular matrix is a square polynomial matrix (i.e., a matrix with entries polynomial powers of s), where the determinant is a nonzero constant. The inverse of such a matrix always exists and it is unimodular itself.*

7.2 Calculating Zeros

The basic idea is to reduce the problem into finding $G_1(s)$, $M(s)$, and $N(s)$, such that

$$G_1(s) = M(s)\ G(s)\ N(s) = \begin{bmatrix} \frac{\epsilon_1(s)}{\psi_1(s)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_2(s)}{\psi_2(s)} & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon_p(s)}{\psi_p(s)} \end{bmatrix} \tag{7.1}$$

where $M(s)$ and $N(s)$ are unimodular matrices, and for each i, polynomials $\epsilon_i(s)$ and $\psi_i(s)$ are relatively prime (i.e., no common roots) with the additional property that

$$\epsilon_1(s) \mid \epsilon_2(s) \mid \epsilon_3(s) \mid \ \ldots \ \mid \epsilon_p(s) \tag{7.2}$$

and

$$\psi_p(s) \mid \psi_{p-1}(s) \mid \ \ldots \ \mid \psi_1(s) \tag{7.3}$$

where $A|B$ denotes 'A divides B', without a remainder. Now we can say: The special form of $G_1(s)$ is called the 'Smith McMillan Form'.

$$Transmission\ Zeros\ of\ G(s)\ = Roots\ of\ \prod_1^p \epsilon_i(s) \tag{7.4}$$

$$Poles\ of\ G(s)\ = Roots\ of\ \prod_1^p \psi_i(s) \tag{7.5}$$

Finding the zeros, through the Smith McMillan forms, can be summarized as follows

- Find the least common denominator of all of the denominators in $G(s)$. Call the resulting monic polynomial $d(s)$.
- Form $G(s) = \frac{1}{d(s)}P(s)$, where $P(s)$ is now a polynomial matrix.
- Through elementary operations - via unimodular matrices - transform $P(s)$ into the 'Smith' form (i.e., the Smith McMillan form with all of the denominators equal to 1).
- Divide diagonal entries by $d(s)$ to the Smith-McMillan form and read off the poles and zeros.
- For this process to be successful, the lowest power polynomial should be placed in the (1,1) element (by elementary operations).

Let us work an example in detail. We start with the following system

$$G(s) = \frac{1}{d(s)}P(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{bmatrix}.$$

We start the elementary operations: First eliminate the (2,1) and (3,1) elements by adding a proper multiple of the first row to the second and third rows

$$P(s) = M_1^{-1}(s)M_1(s)P(s) = M_1^{-1}(s)P_1(s)$$

with

$$P_1(s) = \begin{bmatrix} 1 & -1 \\ 0 & 3(s^2-4) \\ 0 & 3(s^2-4) \end{bmatrix}.$$

Next, we eliminate the (3,2) element

$$P(s) = M_1^{-1}(s)M_2^{-1}(s)M_2(s)P_1(s) = M_1^{-1}(s)M_2^{-1}(s)P_2(s)$$

where

$$P_2(s) = \begin{bmatrix} 1 & -1 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}.$$

Next, we eliminate the (1,2) element

$$P(s) = M_1^{-1}(s)M_2^{-1}(s)P_2(s)N_1(s)N_1^{-1}(s) = M_1^{-1}M_2^{-1}(s)P_3(s)N_1^{-1}(s)$$

where

$$P_3(s) = \begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}.$$

Lastly, we scale the second column through

$$\begin{aligned} P(s) &= M_1^{-1}(s)M_2^{-1}(s)P_3(s)N_2(s)N_2^{-1}(s)N_1^{-1}(s) \\ &= M_1^{-1}(s)\,M_2^{-1}(s)\,P_4(s)N_2^{-1}(s)\,N_1^{-1}(s) \end{aligned}$$

where

$$P_4(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s^2-4) \\ 0 & 0 \end{bmatrix}.$$

Note that

$$G(s) = \frac{1}{d(s)}P(s) = M_1^{-1}(s)M_2^{-1}(s)\frac{1}{d(s)}P_4(s)N_2^{-1}(s)N_1^{-1}(s)$$

or equivalently, from the definition, pre and post multiplying with $M_i(s)$ and $N_i(s)$,

$$G_1(s) = \frac{1}{d(s)}P_4(s), \quad M(s) = M_2(s)M_1(s), \quad N(s) = N_1(s)N_2(s).$$

As a result, we have poles at (-1,-1,-2) and zeros at (+2) (nonminimum phase zero).

7.3 Zeros and the State Space Approach

In this subsection, we will discuss finding transmission zeros from the state space representation of a system. Consider a system with

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{7.6}$$

Associated with this system, let us introduce the following 'system matrix'

$$S(s) = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \tag{7.7}$$

Assumption 7.4. *The system is not* degenerate*, where by degenerate we mean the rank of S is strictly less than the minimum of $[n + rank(B)\ ,\ n + rank(C)]$ for all values of s. If the system is degenerate, one has to be careful. Most codes fail in such cases. This condition (i.e., nondegenerate) is often expressed as S having a normal rank equal to the minimum of $[n + rank(B)\ ,\ n + rank(C)]$.*

Definition 7.5. *Invariant zeros are those values of s that result in rank of S becoming less than the minimum of $[n + rank(B)\ ,\ n + rank(C)]$.*

Remark 7.6. *If there is direct feedthrough, i.e., $y = Cx + Du$, simply change the '0' in the lower right corner of $S(s)$ to D. The rest follows as before.*

Remark 7.7. *If the system is both controllable and observable (and non-degenerate), the invariant zeros are the same as the transmission zeros, discussed earlier. Indeed, there are only one kind of zeros. They can be calculated, for example if rank of B is smaller, by studying $Sx = 0$ and turning it into a generalized eigenvalue problem of the form $Mx = \lambda Nx$. Then standard eigenvalue problems can be used to solve for the eigenvalues (which are the invariant zeros).*

When the system is either uncontrollable or unobservable, the problem become more complicated. For example, let (A, C) have an unobservable mode; i.e., there exists a nonzero vector x such that $[Ax = \lambda x\ ,\ Cx = 0]$. Then this λ is known as an 'output decoupling' zero. 'Input' decoupling zeros are defined similarly. In such cases, system zeros are the transmission zeros plus the decoupling zeros.

Remark 7.8. *Decoupling zeros sometimes are invariant zeros, sometimes not!! Therefore, one has to be careful (see Homework Problems).*

Remark 7.9. *It is possible not to have any zeros (e.g., $\frac{1}{s^2}$).*

Remark 7.10. *Let there be m inputs, and l outputs. Then we have the following*

$$\begin{array}{l} number\ of\ zeros\ \leq n - \max(m,l)\ \ if\ \ D = 0 \\ number\ of\ zeros\ \leq n\ \ if\ \ D \neq 0 \\ number\ of\ zeros\ \leq n - m - d\ \ if\ \ D = 0\ \ m = l\ \ and \\ d = rank\, def\, ficiency of(CB) \end{array}$$

Remark 7.11. *Let the number of outputs be at least as large as the number of inputs (i.e., $l \geq m$), then*

$$\begin{aligned} i)\ squaring\ down\ \ &:\ \ s\ a\ zero\ of \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \\ &\Rightarrow\ \ s\ a\ zero\ of \begin{bmatrix} sI - A & -B \\ FC & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} ii)\ squaring\ up\ \ &:\ \ s\ a\ zero\ of \begin{bmatrix} sI - A & -BG \\ C & 0 \end{bmatrix} \\ &\Rightarrow\ \ s\ a\ zero\ of \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} iii)\ s\ \ \ a\ \ \ zero\ of &\begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \\ \Rightarrow\ \ s\ a\ zero\ of &\begin{bmatrix} sI - A - BKC & -B \\ KC & 0 \end{bmatrix} \end{aligned}$$

7.4 PROBLEM SET

Exercise 7.12. *In the example used earlier, find the unimodular matrices, $M_1(s)$, $M_2(s)$, $N_1(s)$, $N_2(s)$, and their inverses. Finally, what are $M(s)$ and $N(s)$ that turn the transfer function into the Smith form?*

Exercise 7.13. *Using the Smith McMillan form, find the poles and zeros of the following transfer function $G(s)$*

$$\frac{1}{(s+1)(s+2)(s-1)} \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix}$$

Exercise 7.14. *Consider the following system*

$$S(s) = \left[\begin{array}{ccc|c} s-1 & 0 & 0 & 0 \\ 0 & s+1 & 0 & 1 \\ 0 & 0 & s+3 & 1 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

Is this system controllable? Is it observable? Is there an input decoupling zero which is not an invariant zero?

Exercise 7.15. *Show the properties of the last remark are true. What are the corresponding results for the case of $m \geq l$?*