8 CONTROL SYSTEM DESIGN

8.1 Single Input Single Output Systems

For the SISO system shown in Figure 8.1, we use the standard notation:

- $r(s)$: The input command $(r(t))$
- $K(s)$: The Compensator
- $G(s)$: The plant
- $u(s)$: The control or command $(u(t))$
- $y(s)$: The output $(y(t))$
- $d(s)$: The disturbance signal $(d(t))$
- $n(s)$: The sensor noise $(n(t))$
- $e(s)$: The error signal $(e(t))$

Throughout this Section, we will the concern ourselves with the set up of Figure 8.1: i.e., the so called one degree of freedom framework. For the two degree of freedom approach (which yield similar but not identical results), consult the textbook by Wolovich. Also, the same text can be considered an excellent source for material covered in this Section. Simple manipulations yield the following

$$y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} [r(s) - n(s)] + \frac{1}{1 + G(s)K(s)} d(s) \tag{8.1}$$

Definition 8.1. *For a variety of reasons, some of which will be discussed presently, we will use a few transfer functions repeatedly. These are:*

$$1 + G(s)K(s) = Return\ Difference\ Transfer\ Function \tag{8.2}$$

$$S(s) = \frac{1}{1 + G(s)K(s)} = Sensitivity\ Transfer\ Function \tag{8.3}$$

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = Complimentary\ Sensitivity\ Trans.\ Func. \tag{8.4}$$

Note that the complimentary transfer function is simply the closed loop transfer function and $G(s)K(s)$ is just the feedforward loop gain. Also, note that $T(s) + S(s) = 1$.

For a good control system design, a variety of issues are taken into consideration. Let us consider a few of the more important ones

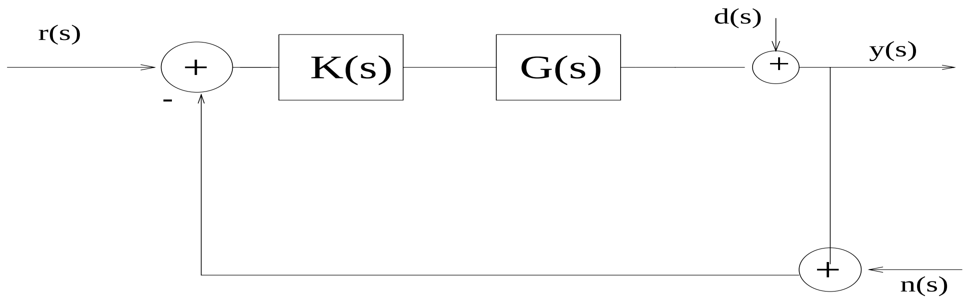

Figure 8.1: Block Diagram of a SISO System

1. Sensitivity to Modeling Error
2. Command Following
3. Disturbance Rejection
4. Noise Propagation
5. Stability Robustness

Clearly, this is not a complete list (leaves transient response out, for example), but it does cover most of the traditionally critical issues (or design specifications) faced in control system design. Also, the ordering is somewhat misleading. Naturally, stability is always the first priority of any control design.

I. Sensitivity to Modeling Error

Let us assume that the actual plant is of the form

$$G(s) = G^*(s) + \delta G(s)$$

where (*) does not mean adjoint! It simply means ideal, and $\delta G(s)$ reflects the error between the actual and nominal plants.

(i) open loop: The open loop output is

$$y(s) = G(s)\, r(s) = G^*(s)\, r(s) + \delta G(s) r(s) = y^*(s) + \delta y(s)$$

where $y^* = G^* r$ is output of the nominal plant and δy is the error between the nominal and actual outputs. Note that the normalized error has the following from

$$\frac{\delta y(s)}{\delta G(s)} = r(s) \quad or \quad \frac{\delta y(s)}{y^*(s)} = \frac{\delta G(s)}{G^*(s)} \tag{8.5}$$

(i) closed loop: For the closed loop, we have the following . Note that the dependence on 's' is dropped for brevity

$$y = \frac{GK}{1+GK}\, r = \frac{(G^* + \delta G)K}{1 + (G^* + \delta G)K}\, r = y^* + \delta y$$

where y^* is the closed loop response of the ideal -nominal - system; i.e.,

$$y^* = \frac{G^* K}{1 + G^* K}\, r$$

and $\delta y = y - y^*$. Next,

$$1 + \frac{\delta y}{y^*} = \frac{y}{y^*} = \frac{G(1 + G^* K)}{(1 + GK)G^*}.$$

As a result, after simplification, we have

$$\frac{\delta y}{y^*} = \frac{\delta G}{G^*} . \frac{1}{1 + GK} \tag{8.6}$$

Remark 8.2. *The error in the output, due to error in modeling, is modified by the sensitivity transfer function (compare* (8.5) *with* (8.6)*). As a result, for the frequencies where δG is 'large', relatively speaking, a small sensitivity (or a 'large' GK) can be used to reduce the closed loop sensitivity (compared to the open loop sensitivity). Notice that we are not discussing stability - only sensitivity.*

II. Command Following

To study command following, set the other inputs to zero; i.e., $n = d = 0$. We then have

$$y = \frac{GK}{1 + GK}\, r$$

If $(1 + GK)$ is 'large', so is - or must be - GK. Therefore, their ratio is approximately one.

Remark 8.3. *Over the frequencies where following r is desired, we want y to be close to r, or the closed loop transfer function be close to one. As a result, 'large' GK is needed. Often, the command signals are in the low frequencies.*

III. Disturbance Rejection

Let $n = r = 0$, to focus of effects of d on the output. We then have

$$y = \frac{d}{1 + GK} \quad or \quad e = \frac{-1}{1 + GK}\, d$$

Remark 8.4. *Over the frequencies where there are large disturbances, it is desirable to have GK large, to reduce their effects . In most cases, such disturbances are in the low frequencies .*

IV. Effects of Noise

We set $r = d = 0$, to focus on noise (n) in the output measurements. It is easy to see that

$$y = -\frac{GK}{1+GK}\, n$$

In this case, note that we want the effects of noise on the output to be small. Note that this requires that GK be small.

Remark 8.5. *Over the frequencies where there is significant sensor noise (typically high frequencies), we need 'small' GK. As a result, 'small' GK is needed.*

V. Stability and Robustness

Naturally, we require nominal stability (i.e., the nominal closed loop is stable). This is the minimum requirement, of course. The more interesting issue is the stability of closed loop system is the actual model is not the same as the nominal one.

Let the plant be in the following form

$$G(s) = G^*(s)\ [1 + L(s)]$$

which is called the form with the multiplicative uncertainty . Note that one can always use $L(s) = \frac{\delta G(s)}{G^*(s)}$. We want to study the stability of the closed loop system of the actual plant, for a range of $L(s)$. Since we will assume very little knowledge of $L(s)$, we will rely on the Nyquist plots. To this end, we will assume the following

Assumption 8.6. *The number of unstable poles of $G(s)$ - the actual plant - is the same as the number of unstable poles of the nominal plant $G^*(s)$.*

Assumption 8.7. *An upper bound for the magnitude of the uncertainty is available; i.e, a function $l(w)$ is available such that $|L(jw)| \leq l(w)$. No other assumptions is made regarding $L(s)$.*

Assumption 8.8. *The controller, $K(s)$ is chosen so that the nominal closed loop system is stable.*

Assumption 8.6 is needed so that the encirclement count of the Nyquist plots remains constant over all allowable uncertainty. Recall the following from the standard SISO systems.

Definition 8.9. *The closed loop system is stable if $K(jw)G(jw)+1 \neq 0$ and the encirclements (of plots of $K(jw)G(jw)$) around $(-1, 0)$ point (on the complex plain) is not changed -for all allowable $L(s)$- from that of the nominal system (i.e., $K(jw)G^*(jw)$).*

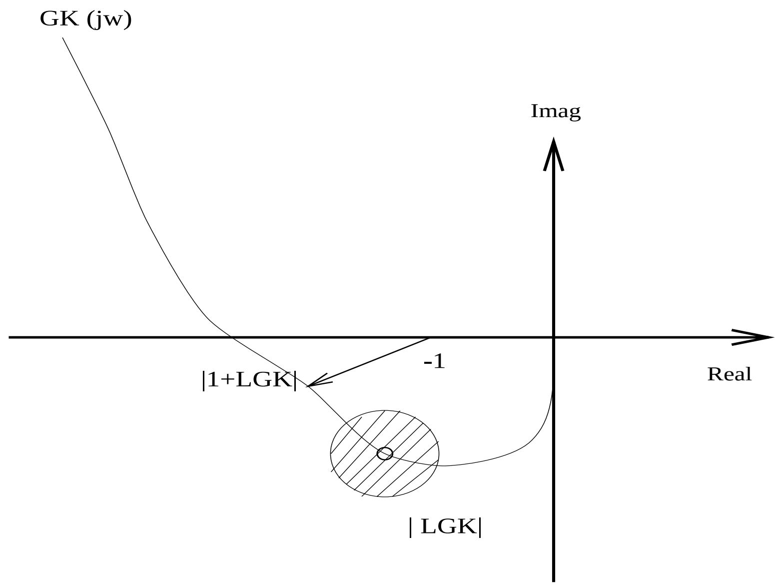

Figure 8.2: Nyquist plot for multiplication uncertainty

One way, to guarantee that a change in the encirclement count is avoided, is by noting that we need

$$1 + K(jw)G^*(jw) + K(jw)G^*(jw)L(jw) \neq 0$$

and thus by requiring the following

$$|L(jw)G^*(jw)K(jw)| \leq |1 + G^*(jw)K(jw)| \tag{8.7}$$

which is satisfied if the following holds

$$|L(jw)| \leq l(w) \leq \frac{|1 + G^*(jw)K(jw)|}{|G^*(jw)K(jw)|} \tag{8.8}$$

or

$$\frac{|K(jw)G^*(jw)|}{|1 + K(jw)G^*(jw)|} < \frac{1}{l(w)}. \tag{8.9}$$

Remark 8.10. *In practice, good models can be obtained over low frequencies. As a result, typical form of $l(w)$ is a curve that grows and becomes large as w increases. On large frequencies, therefore, the nominal transfer function must be very small, as a result, small KG^* are desired over high frequencies (or frequencies where large modeling errors exist).*

As a result of the above discussion, we arrive at the typical shape that we expect the Bode plot to have (see Figure 8.3). The exact cut-offs and lower and higher bound, of course, depend on the exact specifications of the control system.

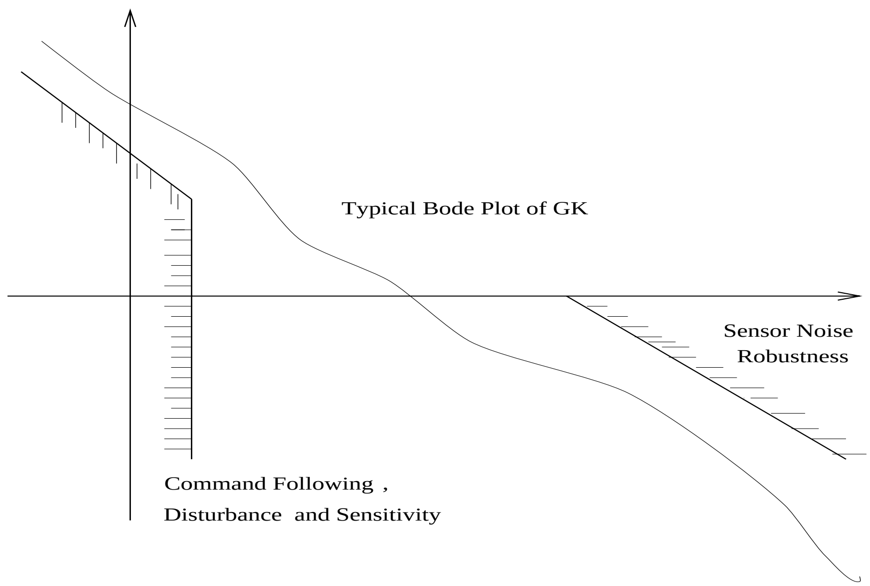

Figure 8.3: A typical desirable Bode Plot

8.2 Multi Input Multi Output Systems

We go parallel to the SISO case (as far as we can). The block diagram will look the same, for example. We will use upper case letters for signals $U(s)$, $Y(s)$, etc. to underline the fact that they are vectors. We start by using

$$E(s) = R(s) - Y(s) - N(s) \;\; and \;\; Y(s) = D(s) + G(s)K(s)E(s)$$

Remark 8.11. *In MIMO systems, $K(s)G(s) \neq G(s)K(s)$, and you need to be careful. Typically, they appear in reverse order of the block diagram; i.e.*

$$U(s) = K(s)E(s) \;\; and \;\; Y(s) = G(s)U(s) + D(s) = G(s)K(s)E(s) + D(s)$$

Next, we have

$$Y(s) = D(s) + G(s)K(s)R(s) - G(s)K(s)Y(s) - G(s)K(s)N(s)$$

or solving for $Y(s)$

$$Y(s) = [I+G(s)K(s)]^{-1}G(s)K(s)[R(s)-N(s)]+[I+G(s)K(s)]^{-1}D(s) \quad (8.10)$$

where similar to the SISO systems we can define

$$I + G(s)K(s) = Return\ Difference\ Transfer\ Function\ (at\ the\ output) \quad (8.11)$$

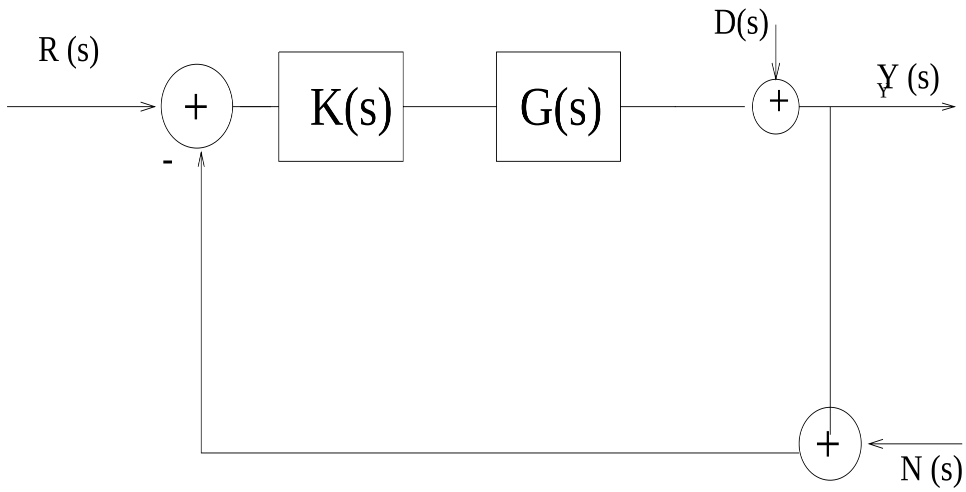

Figure 8.4: Block Diagram of a MIMO System

$$S(s) = [I + G(s)K(s)]^{-1} = Sensitivity\ Transfer\ Function\ (at\ the\ output) \tag{8.12}$$

$$\begin{aligned} T(s) &= [I + G(s)K(s)]^{-1} G(s)K(s) \\ &= Complimentary\ Sensitivity\ Transfer\ Function \end{aligned} \tag{8.13}$$

As before, it is easy to see that $T(s)$ is the closed loop transfer function and

$$T(s) = S(s)\,G(s)\,K(s), \qquad T(s) + S(s) = I \tag{8.14}$$

Similar to the SISO case, we will study the following issues

1. Sensitivity to Modeling Error
2. Command Following
3. Disturbance Rejection
4. Noise Propagation
5. Control Effort
6. Stability Robustness

Before going through each case, let us discuss the notion of ‘large’ and ‘small’ for MIMO systems. In the SISO case, everything was relatively simple; we took the absolute value of the transfer function, evaluated on the imaginary axis.

That is, we looked at $|G(jw)|$, which is consistent with standard notion of Bode plots (input at that frequency gets magnified by the size of the transfer function).

For MIMO system, we also consider what the transfer function does to the incoming signal. We call a transfer function 'large' if it magnifies the amplitude of signal, and 'small' if it does the reverse. The problem is that our inputs and outputs are vectors themselves.

From now on, a signal is considered large if its 2-norm (in frequency or time domain - recall Parseval's Theorem) is large. Therefore, we are interested in seeing what does a given transfer function do to the 2-norm of its input. Now considering the basic properties of the singular value, we have

$$\begin{aligned} Y(jw) &= G(jw)U(jw) \Rightarrow \underline{\sigma}[G(jw)]\ \|U(jw)\|_2 \leq \|Y(jw)\|_2 \\ &\leq \overline{\sigma}[G(jw)]\ \|U(jw)\|_2 \end{aligned}$$

where the maximum (or minimum) is over all frequencies. A 'small' transfer function, therefore, has a small maximum singular value and a 'large' transfer function has a large minimum singular value. These singular values indicate the best and worst case magnification a signal (at a given frequency) may experience. Also, due to Parseval's Theorem, these principle gains can be interpreted at L_2 gains for the convolutions (e.g., given a signal with unit energy what are the upper and lower bounds on the energy of the output).

With this in mind, we can go over the design issues listed above.

I. Sensitivity to Modeling Error

Let us assume that the actual plant is of the form

$$G(s) = G^*(s) + \Delta G(s)$$

where $G^*(s)$ is the nominal or ideal plant , and $\Delta G(s)$ reflects the error between the actual and nominal plants. The following can be shown relatively easily (see the Problem Set!)

(i) open loop

$$\Delta Y(s) = \Delta G(s)\ [G^*(s)]^{-1}\ Y^*(s) \tag{8.15}$$

(i) closed loop

$$\Delta Y(s) = [I + G(s)K(s)]^{-1}\ \Delta G(s)\ [G^*(s)]^{-1}\ Y^*(s) \tag{8.16}$$

Remark 8.12. *The error in the output, due to error in modeling, is modified by the sensitivity transfer function (compare* (8.15) *with* (8.16)*. As a result, for the frequencies where ΔG is 'large', we have large open loop sensitivity. To reduce this sensitivity, the sensitivity transfer function can be exploited.*

To reduce sensitivity, we require the maximum singular value of $S(s)$ be small. Recalling the property that $\overline{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})}$, we arrive at the requirement that minimum singular value of the return difference transfer function be large. Recalling the property that $\underline{\sigma}(I + A)$ is between $(\underline{\sigma}(A) - 1)$ and $(\underline{\sigma}(A) + 1)$, we conclude that $\underline{\sigma}(G(s)K(s))$ should be large, over frequencies of interest (i.e., where we want little sensitivity). Often, such a requirement is written as requiring

$$\underline{\sigma}[G(jw)K(jw)] \geq p(w). \tag{8.17}$$

The function $p(w)$ can thus be used to set the levels and ranges of the frequencies of interests (e.g., $p(w)$ large at low frequencies, to ensure little sensitivity to modeling errors in such a range).

II. Command Following

As before, to study command following, set the other inputs to zero; i.e., $N = D = 0$. We then have

$$Y(s) = T(s)\ R(s)\ .$$

Therefore for good command following over the frequencies of interest (i.e., most often low frequencies), we need $T(s)$ be approximately identity. Since $I = T(s) + S(s)$, this requires very small $S(s)$. The rest follows as before and we end up with requirement of high minimum singular value for $G(jw)K(jw)$.

Remark 8.13. *Over the frequencies where following R is desired (i.e, typically low frequencies), large $\underline{\sigma}[GK]$ is desired.*

III. Disturbance Rejection

Let $N = R = 0$, to focus of effects of D on the output. Considering (8.10), to reduce the effects of disturbance on the output, $S(s)$ should be small, over the range of disturbance frequencies. Similar to the sensitivity analysis, we will need large $\underline{\sigma}[G(jw)K(jw)]$, for the range of frequencies with large disturbance (typically, low frequencies).

Next, we consider cases that may require small $\overline{\sigma}[G(jw)K(jw)]$.

IV. Effects of Noise

We set $R = D = 0$, to focus on noise (N) in the output measurements. Again, we examine (8.10). We have seen that large $\underline{\sigma}[GK]$ results in $T(s) \approx I$, which is certainly undesirable for those frequencies with large noise component. To reduce noise, we need to make $T(s)$ as small as possible (over these noisy frequencies). That is, we want $\overline{\sigma}[T(jw)]$ be very small. Since $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B)$, we have

$$\overline{\sigma}[T(jw)] \leq \overline{\sigma}[S(jw)]\overline{\sigma}[G(jw)K(jw)] = \frac{\overline{\sigma}[G(jw)K(jw)]}{\underline{\sigma}[I + G(jw)K(jw)]} \tag{8.18}$$

Since $\underline{\sigma}(I + A)$ is between $(\underline{\sigma}(A) - 1)$ and $(\underline{\sigma}(A) + 1)$, the fraction on the right hand side of (8.18) can be made small if we make $\overline{\sigma}[G(jw)K(jw)]$ very small (see the Problem Set).

Remark 8.14. *Over the frequencies where there is significant sensor noise (typically high frequencies), we need 'small' GK. As a result, small $\overline{\sigma}[GK]$ is needed.*

V. Control Energy

One can show that

$$U(s) = K(s)S(s)[R(s) - N(s) - D(s)] \tag{8.19}$$

So to reduce control energy, one would seek small $\overline{\sigma}[K(s)S(s)]$. Dropping the dependency of 's' for brevity, we can rewrite this as

$$\overline{\sigma}[K[I + GK]^{-1}] \leq \overline{\sigma}(K)\overline{\sigma}[I + GK]^{-1} \leq \frac{\overline{\sigma}(K)}{\underline{\sigma}[I + GK]}.$$

The goal now is to reduce the RHS of the above equation, which implies needing

$$\underline{\sigma}[I + GK] \gg \overline{\sigma}(K).$$

Now, since $\underline{\sigma}[I + GK] \leq \overline{\sigma}[I + GK] \leq I + \overline{\sigma}(K)\overline{\sigma}(G)$, we must be seeking

$$I + \overline{\sigma}(K)\overline{\sigma}(G) \gg \overline{\sigma}(K) \quad \textit{or} \quad \frac{1}{\overline{\sigma}(K)} + \overline{\sigma}(G) \gg 1. \tag{8.20}$$

Now consider two regions and the typical behavior of $G(jw)$ in these regions:
a) Low Frequencies: $G(jw)$ is large, which implies (8.20) can be realized.
b) High Frequencies: $G(jw)$ is small, which implies (8.20) can be realized if $\overline{\sigma}(K)$ is very small. Since $\overline{\sigma}(GK) \leq \overline{\sigma}(G)\overline{\sigma}(K)$, the need for low control energy requires a low $\overline{\sigma}(GK)$ over high frequencies.

In summary, for small control effort, the maximum singular value of GK should be small at the high frequencies. Naturally, similar results could have been obtained in the SISO case, as well.

Remark 8.15. *A similar result can be obtained by the following: Assume square and invertible G and K, and show that high $\underline{\sigma}(GK)$ results in an approximate relationship $U \approx G^{-1}[R - N - D]$. Therefore, any reference, disturbance or noise at high frequency results in very high control energy.*

VI. Stability Robustness

As you might expect, we will be following the development of the SISO systems. First, however, we need to review the MIMO version of the Nyquist criterion

Definition 8.16. *$G(s)$ is stable if the number of the encirclements (around the origin) of the map $det[I + G(jw)K(jw)]$, evaluated on the Nyquist D-contour, is equal to the negative of the number of unstable open-loop modes (poles) of $G(jw)$.*

We start by assuming the following relationship between the actual plant (i.e., $G(s)$), and nominal plant (i.e., $G^*(s)$).

$$G(s) = [I + L(s)]\ G^*(s)\ . \tag{8.21}$$

We are modeling the uncertainty as multiplicative *at the output*. Note that for MIMO systems, at the output and at the input are not the same. As with the SISO system, we make the following assumptions:

Assumption 8.17. *The nominal closed loop system is stable (meets the MIMO Nyquist criterion and determinant of $I + G^*K]$ is not zero at any frequency) and the number of unstable poles $G(s)$ - the actual plant - is the same as the number of unstable poles of the nominal plant $G^*(s)$.*

Assumption 8.18. *An upper bound for the magnitude of the uncertainty is available; i.e, a function $l(w)$ is available such that*

$$\overline{\sigma}[L(jw)] \leq l(w) \qquad \forall w.$$

.

As before, we will simplify notation by dropping 'jw'. For stability, the assumptions above imply that it is sufficient to have $det[I + (I + L)G^*K] \neq 0$ for all allowable $L(jw)$. Therefore, we need

$$\underline{\sigma}[I + (I + L)G^*K] > 0 \qquad \forall w\ . \tag{8.22}$$

Note, however, that

$$I + (I + L)G^*K = I + G^*K + LG^*K = [I + LG^*K(I + G^*K)^{-1}](I + G^*K)$$

assuming the inverse exist (Not a big assumption: nominal stability). Next,

$$\underline{\sigma}[I + (I + L)G^*K] \geq \underline{\sigma}(I + G^*K)\ \underline{\sigma}[I + LG^*K(I + G^*K)^{-1}]$$

where, by nominal stability, the first term on the right is nonzero. As a result, we focus on the second term and see if we can make it nonzero. Clearly (see Problem Set), this term is nonzero if we require

$$\underline{\sigma}(I) = 1 > \overline{\sigma}[LG^*K(I + G^*K)^{-1}]$$

or equivalently if

$$1 > \overline{\sigma}(L)\overline{\sigma}[G^*K(I + G^*K)^{-1}] \geq \overline{\sigma}[LG^*K(I + G^*K)^{-1}]$$

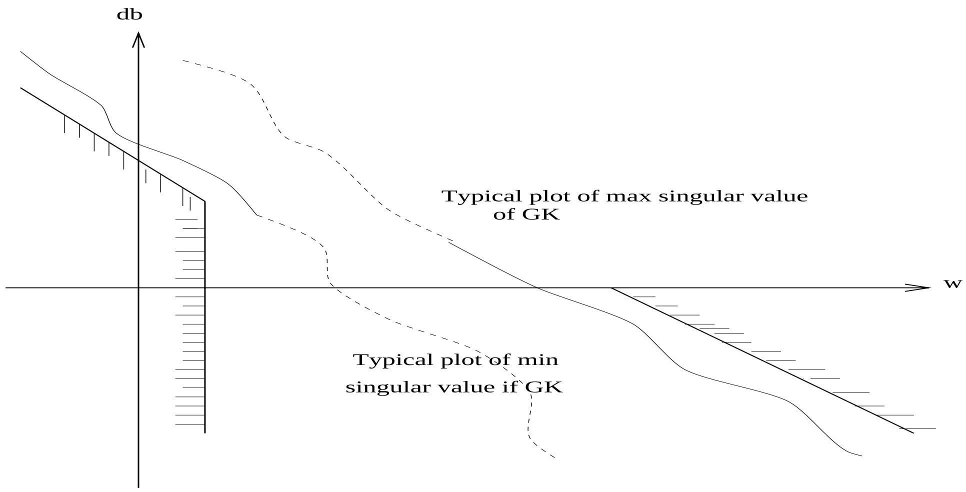

Figure 8.5: Max and Min singular value plots

Since $l(w) \le \overline{\sigma}(L)$, then (8.22) holds if

$$1 > l(w)\overline{\sigma}[G^*K(I + G^*K)^{-1}] \ge \overline{\sigma}[LG^*K(I + G^*K)^{-1}]$$

or

$$\frac{1}{l(w)} > \overline{\sigma}[G^*K(I + G^*K)^{-1}]. \qquad (8.23)$$

So far so good. Since in high frequencies, we have $l(w) \gg 1$, stability robustness - i.e., (8.23) - dictates a small maximum singular value for the nominal transfer function (note the very small trick involved). From previous development, we know that this requirement is the same as having small $\overline{\sigma}(GK)$ over high frequencies.

As a result of the above development, we arrive at the plots in Figure 8.5 for the max and min singular values of GK. Finally, before we leave this Section, we can study a few interesting issues:

1. Invert (8.23) to get

$$l(w) < \frac{1}{\overline{\sigma}[G^*K(I + G^*K]^{-1}]} = \underline{\sigma}[I + (G^*K)^{-1}]. \quad (why?) \qquad (8.24)$$

The right hand side, therefore, defines the amount of modeling error a given nominal closed loop system can tolerate. In can be interpreted as the MIMO version of gain margin!!

2. What about performance in presence of uncertainty?

Earlier, we discussed sensitivity and model following, and other low frequency requirements in terms of $G(s)$, the actual or nominal. In many applications this may be acceptable since most modeling errors occur in high frequencies. However, if there were low frequency model error; i.e., $l(w)$ was not zero at low frequencies, we need to review the results.

Equation (8.17) implies that

$$\underline{\sigma}[I + G(jw)K(jw)] \geq p(w) \qquad G(jw) = (I + L(jw))G^*(jw)$$

while by (8.23)

$$\overline{\sigma}[G^*K(I + G^*K)^{-1}] < \frac{1}{l(w)}.$$

Now assume that $l(w) \leq 1$ in the low frequencies (i.e., nonzero, but moderate levels of uncertainty at low frequencies). Then

$$p(w) \leq \underline{\sigma}[I + (I + L)G^*K] = \underline{\sigma}\{[I + LG^*K(I + G^*K)^{-1}][I + G^*K]\} \quad (8.25)$$

using $\underline{\sigma}(AB) \geq \underline{\sigma}(A)\underline{\sigma}(B)$, (8.25) is satisfied if

$$p(w) \leq \underline{\sigma}[I + LG^*K(I + G^*K)^{-1}]\underline{\sigma}[I + G^*K]. \quad (8.26)$$

Now, if we assume that the nominal system has high gains at low frequencies, we can use $\underline{\sigma}[G^*K] \gg 1$ in low frequencies, which implies $\underline{\sigma}[I+G^*K] \approx \underline{\sigma}[G^*K]$, with G^*K nonsingular and $T(jw) \approx I$. Equation (8.26) is therefore equivalent to

$$p(w) \leq \underline{\sigma}(I + L)\ \underline{\sigma}(G^*K). \quad (8.27)$$

Since $\underline{\sigma}(AB) \geq \underline{\sigma}(A) - \overline{\sigma}(B)$, (8.27) must be satisfied if

$$p(w) \leq (1 - l(w))\ \underline{\sigma}[G^*K]$$

or

$$\frac{p(w)}{1 - l(w)} \leq \underline{\sigma}[G^*K] \quad (8.28)$$

that is, the modeling uncertainty at low frequencies makes the job of the controller (for performance type measures) more difficult.

8.3 A Few Matrix Identities

Let A and C be nonsingular matrices of possibly different dimensions. If there are, possibly nonsquare, matrices B and D such that BCD is the same dimension as A, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \quad (8.29)$$

The matrix inversion lemma can be applied to a variety of cases. Some, exploit the fact that the dimension of the matrices that require inversion may be smaller on the right hand side. Consider the following - where matrices G_1, G_2, etc. can be either constant of functions of time or 's''

$$\begin{aligned} [I_n + G_2\,G_1\,H_2\,H_1]^{-1}G_2\,G_1 &= G_2[I_m + G_1H_2H_1G_2]^{-1}G_1 && (8.30)\\ &= G_2G_1[I_r + H_2H_1G_2G_1]^{-1} && (8.31)\\ &= G_2G_1 - G_2G_1H_2[I_p + H_1G_2G_1H_2]^{-1}H_1G_2G_1 \end{aligned}$$

$$(P^1 + KC)^{-1} = P - PK(I + CPK)^{-1}CP \tag{8.32}$$

$$(I + KCP)^{-1} = I - K(I + CPK)^{-1}CP \tag{8.33}$$

$$(I + PKC)^{-1} = I - PK(I + CPK)^{-1}C \tag{8.34}$$

$$(I + G)^{-1} + (I + G^{-1})^{-1} = I \quad for\ G\ square\ and\ invertible \tag{8.35}$$

$$\overline{\sigma}[(I + G)^{-1}] + \overline{\sigma}[(I + G^{-1})^{-1}] \geq 1 \tag{8.36}$$

$$\overline{\sigma}[(I + G)^{-1}] + 1 \geq \overline{\sigma}[(I + G^{-1})^{-1}] \tag{8.37}$$

$$\overline{\sigma}[(I + G^{-1})^{-1}] + 1 \geq \overline{\sigma}[(I + G)^{-1}] \tag{8.38}$$

$$\overline{\sigma}(G) \geq \frac{\underline{\sigma}(I + G)}{\underline{\sigma}(I + G^{-1})} \geq \underline{\sigma}(G) \qquad (Use\ \ \underline{\sigma}(AB) \leq \underline{\sigma}(A)\overline{\sigma}(B)\ \ ) \tag{8.39}$$

8.4 PROBLEM SET

Exercise 8.19. *Show (or prove of whatever) properties P1-P3 of singular values.*

Exercise 8.20. *Show (or prove of whatever) properties P4-P13 of singular values.*

Exercise 8.21. *Show or derive equations* (8.15) *and* (8.16).

Exercise 8.22. *Show* $(I + GK)^{-1}GK = I - (I + GK)^{-1}$.

Exercise 8.23. *Show that* $\overline{\sigma}[G(jw)K(jw)] \ll 1$ *implies* $\overline{\sigma}[T(jw)] \ll 1$, *where* $T(s)$ *is the closed loop transfer function.*

Exercise 8.24. *Derive equation* (8.19).

Exercise 8.25. *Let* $A > 0$. *Show that if* $\underline{\sigma}(A) > \overline{\sigma}(B)$, *then* $A + B > 0$.

Exercise 8.26. *Do as many as the matrix identities of the previous section.*