9 LOOP SHAPING AND RECOVERY

In this Section, we will look for ways to obtain desirable loop shapes (as discussed in the previous Section) by the LQR technique. We will also look into effects of observers of the closed loop shapes and ways to 'recover' the desirable loop shapes we can get with LQR, when observers are needed (i.e., we cannot measure the full state).

9.1 The KALMAN Equality

Let us start with the steady state Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{9.1}$$

and the optimal gain matrix

$$u = -Kx, \quad K = R^{-1}B^T P. \tag{9.2}$$

From (9.2), it is clear that the following identities hold

$$R^{\frac{1}{2}}K = R^{-\frac{1}{2}}B^T P \tag{9.3}$$

$$K^T RK = PBR^{-1}B^T P. \tag{9.4}$$

The Riccati equation of (9.1) can be written as

$$PA + A^T P - K^T RK + Q = 0,$$

or after some manipulations

$$P(sI - A) + (-sI - A^T)P + K^T RK = Q. \tag{9.5}$$

Now after multiplying (9.5), on the left, by $R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}$ and, on the right, by $(sI - A)^{-1}BR^{-\frac{1}{2}}$, we will get

$$\begin{aligned}
&R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}P(sI - A)(sI - A)^{-1}BR^{-\frac{1}{2}} \\
&+R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}(-sI - A^T)P(sI - A)^{-1}BR^{-\frac{1}{2}} \\
&+R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}K^T RK(sI - A)^{-1}BR^{-\frac{1}{2}} \\
&= R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}BR^{-\frac{1}{2}} \ ,
\end{aligned}$$

or, considering (9.3) and (9.4),

$$\begin{aligned}
&R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}K^T R^{\frac{1}{2}} + R^{\frac{1}{2}}K(sI - A)^{-1}BR^{-\frac{1}{2}} \\
&+R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}K^T RK(sI - A)^{-1}BR^{-\frac{1}{2}} \\
&= R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}BR^{-\frac{1}{2}} \ .
\end{aligned}$$

The left hand side of this equation is of the form $Y^T + X + Y^T X$. Adding I to both side and noting that $(I+Y)^T(I+X) = I + Y^T + X + Y^T X$, we have

$$[I + R^{\frac{1}{2}} K(-sI - A)^{-1} B R^{-\frac{1}{2}}]^T [I + R^{\frac{1}{2}} K(sI - A)^{-1} B R^{-\frac{1}{2}}] = I + R^{-\frac{1}{2}} B^T (-sI - A^T)^{-1} Q (sI - A)^{-1} B R^{-\frac{1}{2}}. \tag{9.6}$$

Equation (9.6) is known as the Kalman equality (the better known Kalman inequality is very similar; i.e. Left Hand Side (LHS) of (9.6) $\geq I$, for all w, where $s = jw$!).

9.2 Asymptotic behavior of Riccati equation

The book by Kwakernaak and Sivan is the primary source of this section (as well as a paper by B. Francis). Again consider the Riccati equation of (9.1), except we let

$$R = \rho^2 N, \quad \textit{and} \;\; Q = H^T H \tag{9.7}$$

without any loss of generality. We will study the behavior of the positive definite solution of (9.1) as ρ tends to infinity and zero.

Result A: $\rho \to 0$
The limit:

$$\lim_{\rho \to 0} P_\rho = P_o$$

exists. Also, let z_i's, $i = 1, 2, \ldots p$, be the transmission zeros of the system $H(sI - A)^{-1}B$ (which is assumed to be controllable and observable). Then the closed loop poles approach z_i's if z_i is on the closed left half plane (and to $-z_i$ if is on the right half plane). The remaining $(n-p)$ poles go to infinity in a Butterworth pattern.

Lastly, if the transfer function $H(sI - A)^{-1}B$ **is right invertible with no unstable zeros**, then $P_o = 0$. Furthermore, is this case, for some unitary W (i.e, $WW^T = I$)

$$K = R^{-1} B^T P \to R^{-\frac{1}{2}} W H = \frac{1}{\rho} N^{-\frac{1}{2}} W H \tag{9.8}$$

Result B: $\rho \to \infty$
In this case the closed loop pole approach the open loop poles (if stable) or their mirror image with respect to the imaginary axis (if unstable). If the open loop poles are stable, the gain matrix approaches zero.

9.3 Good LQR loop shapes

Recall that for robustness and performance, we would like to shape KG according to some specifications. For MIMO systems, this 'shaping' corresponds to limiting, from below, the minimum singular value of KG or GK over low frequencies and from above, the maximum singular value of KG, over high frequencies. In the following, we will review a method that uses many of the topics we have covered in class.

For simplicity, we make the following assumptions

Assumption 9.1. *Without any loss of generality, we assume that*

$$R = \rho^2 I, \qquad Q = H^T H. \tag{9.9}$$

Recall that for the LQR problem

$$K(s)G(s) = K(sI - A)^{-1}B = K\Phi(s)B \tag{9.10}$$

where

$$K = R^{-1}B^T P = \frac{1}{\rho^2}B^T P, \quad \textit{and} \;\; \Phi(s) = (sI - A)^{-1}. \tag{9.11}$$

Since we are concerned with frequency characteristics of the system, in the remainder if this section we will replace s by jw.

For the particular choice of Q and R used here, the Kalman equality of (9.6) becomes

$$\begin{aligned}[][I \quad &+ \quad K(jwI - A)^{-1}B]^h[I + K(jwI - A)^{-1}B] \\ &= \quad I + \frac{1}{\rho^2}[H(jwI - A)^{-1}B]^h[H(jwI - A)^{-1}B]\end{aligned} \tag{9.12}$$

where *the superscript 'h' denotes complex conjugate transpose*. Using (9.11), we can rewrite (9.12) as

$$[I + KG]^h\,[I + KG] = I + \frac{1}{\rho^2}[H\Phi B]^h\,[H\Phi B] \tag{9.13}$$

which implies

$$\lambda_i\left([I + KG]^h\,[I + KG]\right) = 1 + \frac{1}{\rho^2}\;\lambda_i\left([H\Phi B]^h\,[H\Phi B]\right)$$

which, considering the basic definition of singular values, reduces to

$$\sigma_i([I + KG]) = \sqrt{1 + \frac{1}{\rho^2}\;\sigma_i^2\left([H\Phi B]\right)}. \tag{9.14}$$

Equation (9.14) can be used for low frequency performance. Recall that we typically need very large minimum singular values for KG over these frequencies. Therefore a good approximation from, (9.14), over these low frequencies, is

$$\sigma_{min}(KG) \approx \frac{1}{\rho}\sigma_{min}(H\Phi B). \tag{9.15}$$

As a result, using the free design parameters, i.e., H and ρ, one can increase the minimum singular value. Note that this method is approximate; i.e., equation (9.15) is used to get initial estimates for H and ρ, that satisfy the performance requirements. These estimates are used in the Riccati equation to obtain exact values. (Also, note the inherent trial and error nature). Finally, the exact form of H can be used to make the minimum and maximum singular values closer to one another (particularly over the cross over region).

For robustness, we need to investigate the following conditions

$$\frac{1}{\ell_m(w)} > \overline{\sigma}(I + KG)^{-1}KG, \quad \forall\ w \tag{9.16}$$

or, equivalently, (recall the last Section)

$$\underline{\sigma}(I + [KG]^{-1}) > \ell_m(w). \tag{9.17}$$

From (9.14), it is clear that

$$\underline{\sigma}(I + [KG]) > 1, \quad \forall\ w. \tag{9.18}$$

A few decades ago, Alan Laub showed that (9.18) implies that

$$\underline{\sigma}(I + [KG]^{-1}) > \frac{1}{2},\ \forall\ w \tag{9.19}$$

that is, the LQR controller provides a small robustness bounds for all frequencies automatically. Typically, the bound on the uncertainty ($\ell_m(w)$) becomes larger at high frequencies (considerably larger than 1). In high frequency range, therefore, we need to be concerned with the behavior of the maximum singular value of KG (recall the previous Section).

Now, we will need the following assumption

Assumption 9.2. *The transfer function $H(sI - A)^{-1}B$ is minimum-phase and right invertible.*

From the asymptotic behavior of the Riccati equation, we have (recall that we are using $N = I$ in (9.8))

$$K_c \rightarrow \frac{1}{\rho}WH, \quad as \quad \rho \rightarrow 0 \tag{9.20}$$

where W is an orthogonal matrix. Finally

$$K_cG = K_c(sI - A)^{-1}B = K_c\frac{1}{s}(I - \frac{A}{s})^{-1}B$$

At high frequencies, (i.e., $s = jw$ and w very large), as ρ approaches zero, we can approximate KG by

$$K_cG \longrightarrow \approx \frac{WHB}{jw\rho}, \quad i.e., \quad \overline{\sigma}\,(K_cG) \longrightarrow \approx \overline{\sigma}\,(\frac{HB}{jw\rho}) \tag{9.21}$$

Equation (9.21) can be used for robustness synthesis. Using (9.21) as an approximate value for high frequencies (and small values of ρ), the parameters H and ρ can be sought that satisfy both the performance requirements and the robustness requirements; i.e., keeping the maximum singular value of KG small at high frequencies and minimum singular value of KG large at low frequencies!

While a very small ρ would guarantee the satisfaction of (9.15), it might result in violation of the high frequency requirement (via (9.21)), over some of the intermediate frequencies. A typical consideration is the crossover frequency. The crossover frequencies are defined at those frequencies that result in $\sigma_i(KG) = 1$, for some i. Let w_{cmax} correspond to the maximum singular value of KG becoming one (note that $w_{cmax} \approx \frac{\overline{\sigma}(HB)}{\rho}$). After some thought (and maybe a touch of voo-doo), it becomes clear that this w_{cmax} should be not much larger (or preferably even smaller) than w_l (i.e., the frequency where ℓ_m becomes 1).

Another interesting point is that (9.21) implies that KG rolls off at, approximately, 20 db rate. If $\ell_m(w)$ grows faster than that, the crossover frequencies should be moved to the left.

9.4 Loop Transfer Recovery - LTR

So far, we have seen that LQR can be manipulated so that a desirable loop is obtained. Actually, other interesting results have been proven. For example, the concept of gain and phase margins have been extended to the MIMO case and LQR is shown to have infinite gain margin and sixty degrees phase margin. Due to our time constraint, we will omit these.

In 1978, John Doyle showed that all these nice margins can be lost once the observer is added (i.e., LQG does not necessarily have the same margins as the LQR). Around 1981, Doyle, along with Gunter Stein, followed this line by showing that the loop shape (of GK or KG) itself will, in general, change when the filter is added for estimation . well? What to do? where to go? The basic idea is very simple: suppose the LQR design has resulted in a great loop shape. What can be done (and indeed if) to recover the nice loop shape, if an observer is needed.

9.4.1 The Basic Idea

We will start with the basic system

$$G(s) = C(sI - A)^{-1}B = C\Phi B. \tag{9.22}$$

where, as before,

$$\Phi = (sI - A)^{-1}. \tag{9.23}$$

Without specifying the controller or observer gain matrices, we can represent the full state closed loop systems and the observer-based closed loop system with the state space form; i.e.

Full State:

$$\begin{cases} \dot{x} = Ax + Bu \\ u = -Kx \end{cases} \tag{9.24}$$

Observer-based:

$$\begin{cases} \dot{x} = Ax + Bu \\ \dot{\hat{x}} = A\hat{x} + Bu + F(y - C\hat{x}) = (A - FC)\hat{x} + Bu + Fy \\ u = -K\hat{x} \end{cases} \tag{9.25}$$

Taking the Laplace transform of (9.24) and (9.25) would result in

$$x(s) = \Phi Bu(s) \tag{9.26}$$

$$(sI - A)\hat{x}(s) = Bu(s) + Fy(s) - FC\hat{x} = -(BK + FC)\hat{x}(s) + Fy(s) \tag{9.27}$$

From now on, for brevity, we will use y instead of $y(s)$, etc. The dependence of y, u etc on s will be clear from the context. We can now make a few interesting remarks:

1. Note that the compensator may not be stable!

$$u = -K(\Phi^{-1} + BK + FC)^{-1}Fy = -K\Psi Fy \tag{9.28}$$

where by Ψ we clearly mean

$$\Psi = (sI - A + BK + FC)^{-1}. \tag{9.29}$$

2. We may also consider the LQR controller as a controller designed for general purpose (and not necessarily regulation). So consider Figures 9.1 and 9.2, where the closed loop system is presented in the block diagram form, with potentially nonzero reference input r. Keep in mind that LQR/LQG are designed for the regulation problem, but for other performance specifications (robustness, noise property, etc), the overall shape of the loop might be of some interested. Now, note that the transfer function from the reference to input is the same in full

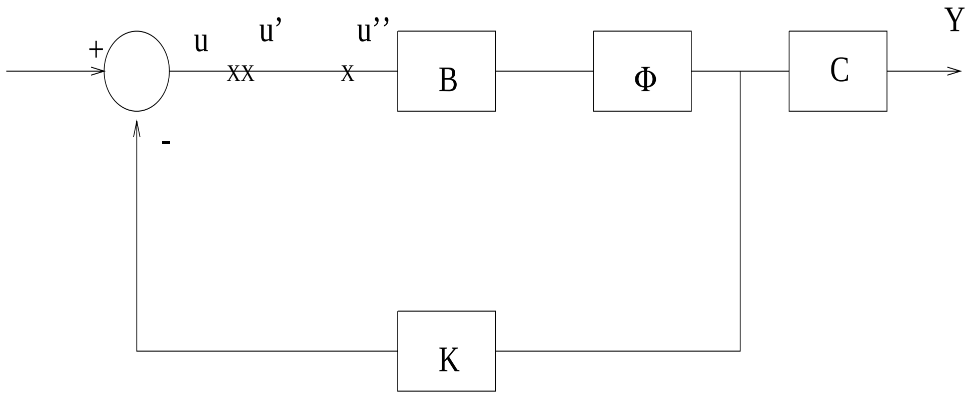

Figure 9.1: Block diagram for the full state controller

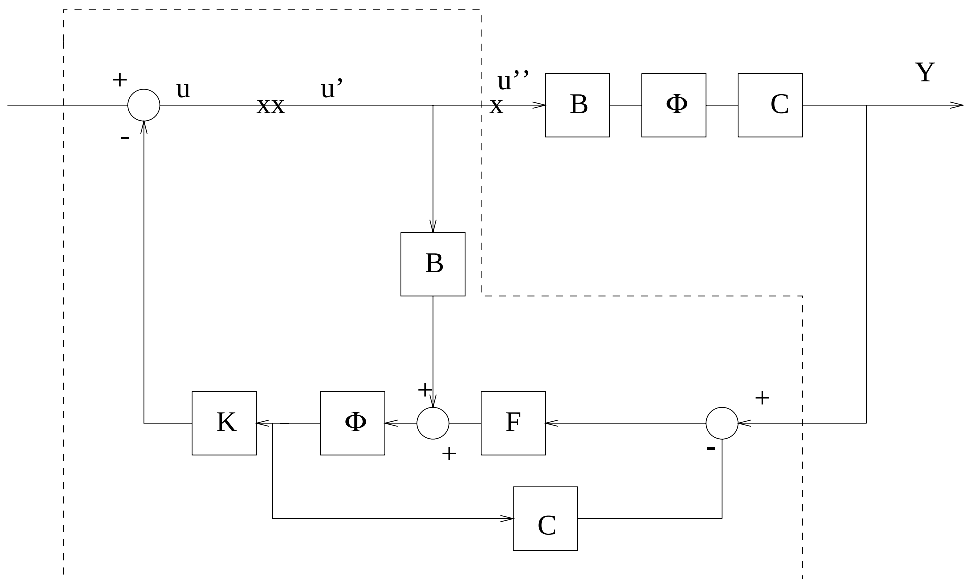

Figure 9.2: Block diagram for the observer based controller

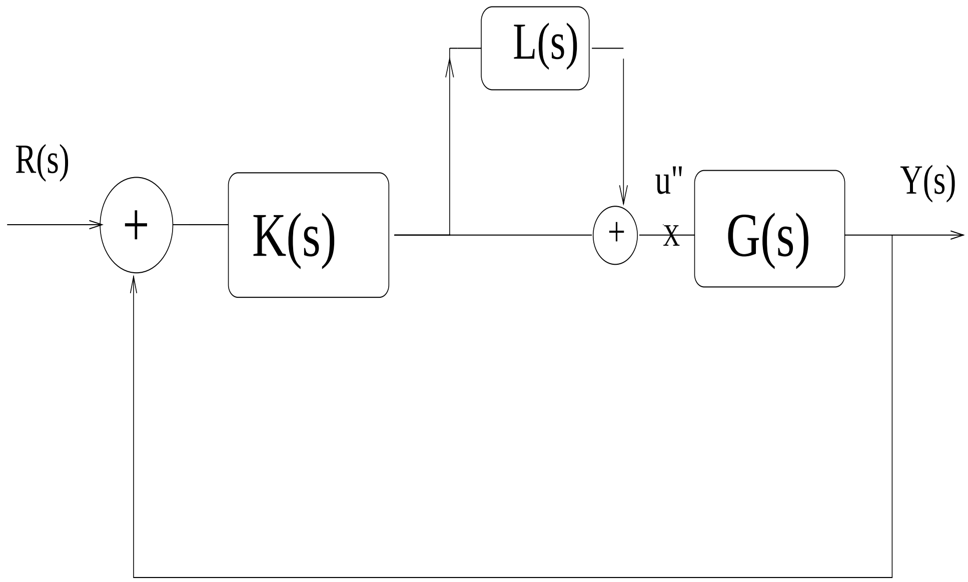

Figure 9.3: Multiplicative uncertainty at the input

state (Figure 9.1) and observer based controllers (Figure 9.2). For the full state case

$$u = r - Kx = r - K\Phi Bu \rightarrow u = (I + K\Phi B)^{-1} r \rightarrow x = \Phi B (I + K\Phi B)^{-1} r \quad (9.30)$$

while for the observer

$$(sI - A + FC)\hat{x} = Bu + FCx = Bu + FC\Phi Bu$$

or

$$(sI - A + FC)\hat{x} = (\Phi^{-1} + FC)\Phi Bu \longrightarrow \hat{x} = \Phi Bu \quad (9.31)$$

the rest follows exactly as the full state case to obtain

$$\hat{x} = \Phi B (I + K\Phi B)^{-1} r. \quad (9.32)$$

From now on, we will focus of the regulator problem and thus $r = 0$. Also, consider Figures 9.1 and 9.2, assume that we will consider 'breaking' the loop at two places; X and XX (the reason for this will become obvious below). Therefore we can consider u the output of the compensator and u' and u'' the input in the plant, depending on where the loop is broken.

Let us discuss the motivation for this. Consider the block diagram of Figure 9.3. The configuration is called 'multiplicative uncertainty at the input', and results in plants of the form $G(I + L)$. Recall that during the compensator design, only the nominal system is used. Information on $L(jw)$, however, is used to design a compensator that accommodates this uncertainty (e.g., the loop shape for KG or GK). Clearly, u and u'' are not the same signal. The first is the output of the compensator and the second is the actual input to the plant (that is u after going through the uncertain dynamics, hence the name of uncertainty at the input!). Furthermore, regardless of the form of controller, the transfer function from u'' to u is exactly KG! (check this yourself). If robustness with respect to uncertainty is your concern, you better make sure the KG (or GK, if appropriate) of the final compensator satisfies the requirements.

Now we can continue with our rambling.

3. In both Figures 9.1 and 9.2, the transfer function from u' to u is $-K\Phi B$. (To see this better, assume that there are unmodeled junk between u and u'. The actual output to the plant is therefore u', while the output of the compensator is u).

$$FullState: \quad u = -Kx = -K\Phi B u'$$

$$Observer: \quad u = -K\hat{x} \stackrel{?}{=} -K\Phi B u'$$

but u' in this case is the plant input and similar to (9.31), we can obtain $\hat{x} = \Phi B u'$. The rest is obvious.

This fact is not particularly interesting! After all, considering Figure 9.2, the point u' is still *inside* the compensator box! A more reasonable way of incorporating unmodeled dynamics is to break the loop outside of the compensator (point X) and consider the transfer function from u'' to u for both controllers.

For the full state feedback, we still have $x = \Phi B u''$, therefore

$$u = -K\Phi B u''. \tag{9.33}$$

For the observe, however, things get rather complicated

$$\hat{x} = \Phi B u + \Phi F C \Phi B u'' - \Phi F C \hat{x} \tag{9.34}$$

taking into account the control law in (9.25)

$$\hat{x} = (I + \Phi F C + \Phi B K)^{-1} \Phi F C \Phi B u'' \tag{9.35}$$

or equivalently

$$u = -K(I + \Phi F C + \Phi B K)^{-1} \Phi F C \Phi B u''. \tag{9.36}$$

Considering the fact that $C\Phi B$ is the plant (i.e., $G(s)$) and u'' the input, we see that $C\Phi B u''$ is nothing but the output. Equation (9.36) is therefore of the form $u = P(s)y$, where P is the compensator. In summary, the transfer function from u'' to u in (9.36) is the 'KG' of the observer based controller! (which, by now, should not surprise you).

Equation (9.36), in general, is not the same as (9.33). To explore the possibility of making them the same, let us rewrite (9.35). We start by rearranging (9.34)

$$(\Phi^{-1} + FC)\hat{x} = Bu + FC\Phi Bu''$$

or

$$\hat{x} = (\Phi^{-1} + FC)^{-1}[Bu + FC\Phi Bu''] \tag{9.37}$$

but, from the matrix inversion lemma, we have

$$(\Phi^{-1} + FC)^{-1} = \Phi - \Phi F(I + C\Phi F)^{-1}C\Phi. \tag{9.38}$$

Simple manipulations show that

$$[\Phi - \Phi F(I + C\Phi F)^{-1}C\Phi]B = \Phi[B - F(I + C\Phi F)^{-1}C\Phi B] \tag{9.39}$$

and

$$\begin{aligned} [\Phi - \Phi F(I + C\Phi F)^{-1}C\Phi]FC\Phi B &= \Phi F[I - (I + C\Phi F)^{-1}C\Phi F]C\Phi B \\ &= \Phi F(I + C\Phi F)^{-1}C\Phi B. \end{aligned} \tag{9.40}$$

HERE COMES THE LIGHTENING! **If** we had

$$F(I + C\Phi F)^{-1} = B(C\Phi B)^{-1} \tag{9.41}$$

from (9.37) to (9.40), the transfer function from u'' to u, via (9.37) would become

$$u = -K\Phi Bu'' \quad !\,!\,!$$

that is , the same as the full state feedback. Choosing F such that (9.41) holds is the central idea of the LTR method. This is not as easy as you might think! To see the difficulty, consider the following. Equation (9.41) implies that

$$\Phi F(I + C\Phi F)^{-1}C\Phi B = \Phi B. \tag{9.42}$$

However, from matrix inversion lemma, we can write

$$\Phi F(I + C\Phi F)^{-1}C\Phi = \Phi - (\Phi^{-1} + FC)^{-1}.$$

For (9.42) to hold, in light of the previous equation, we need

$$(\Phi^{-1} + FC)^{-1}B = 0 \;\; \forall\, w. \longrightarrow (jwI - A + FC)^{-1}B = 0 \;\; \forall w,$$

which is not an easy condition to satisfy. The traditional approaches to this problem end up having observer gains that get larger and larger and satisfy this condition only *asymptotically*.

9.4.2 Observer Design

Recall (9.41). We would like to design an observer such that the gain satisfies

$$F(I + C\Phi F)^{-1} = B(C\Phi B)^{-1}.$$

One way to satisfy this is to have $F = F_q$ (i.e., F a function of scalar q) such that

$$\frac{F_q}{q} \longrightarrow BW \quad , \quad W \ nonsigular. \tag{9.43}$$

In that case

$$F(I + C\Phi F)^{-1} = \frac{F_q}{q} q(I + C\Phi F_q)^{-1} = \frac{F_q}{q}(\frac{I}{q} + C\Phi\frac{F_q}{q})^{-1}$$

and as $q \longrightarrow \infty$, we get

$$F(I + C\Phi F)^{-1} \to BW(C\Phi BW)^{-1} = B(C\Phi B)^{-1}$$

which is (9.41). Therefore, instead of (9.41), we can focus on (9.43). Now consider the Kalman filter equation

$$AS + SA^T + Q_f - SC^T R^{-1} CS = 0, \quad F = SC^T R^{-1} \tag{9.44}$$

Instead of a typical Q_o, use

$$Q_f = Q_o + q^2 BVB^T \tag{9.45}$$

where V can be any positive definite matrix (choose I if you want). The Riccati equation in (9.44) can be written as

$$A\frac{S}{q^2} + \frac{S}{q^2}A^T - q^2\frac{S}{q^2}C^T R^{-1} C\frac{S}{q^2} + BVB^T + \frac{Q_o}{q^2} = 0. \tag{9.46}$$

From basic properties of Riccati equation, **if the nominal system is minimum-phase and left invertible**, then

$$q \to \infty \implies \frac{S}{q^2} \to 0 \tag{9.47}$$

and furthermore, along the lines of equation (9.8)

$$F = SC^T R^{-1} \to qBV^{\frac{1}{2}} U R^{-\frac{1}{2}} = qBW \tag{9.48}$$

for some nonsingular W, as desired. So the method boils down to the following easy to use, NASA approved bullet chart

- Design the LQR such that KG has the appropriate loop shape
- Check loop shape after adding the nominal observer

- Increase q in (9.45) and (9.46), to recover the LRQ loop, pointwise in the frequency domain

Almost everything here can be done with the dual problem. When uncertainty (and disturbances) are best modeled by multiplicative uncertainty at the output, the approach is to design the filter so that GK has the desired shape. Then by manipulating the control Riccati equation recover the loop. We can now present the last bullet chart regarding the short comings of the LTR process

- Requires minimum phase property
- Uncertainty either at the input or at the output
- Can only recover the loop through high gains (i.e., asymptotically)

9.5 PROBLEM SET

Exercise 9.3. *Make sure you can follow the steps from* (9.5) *and* (9.6). *Also, why is the right hand side of* (9.6) *greater than or equal to the identity matrix for* $s = jw$*, for all* w*?*

Exercise 9.4. *show how* (9.6) *implies that* (9.12) *holds.*

Exercise 9.5. *The conditions in* (9.9) *are claimed to cause any loss of generality. Can it be true?*

Exercise 9.6. *Show i:* (9.28) *holds, ii:* (9.39) *and* (9.40) *hold, and iii:* (9.48) *holds.*

Exercise 9.7. *Is the following identity true?*

$$(sI - A + FC)^{-1} = [I + (sI - A)^{-1}FC]^{-1}\ (sI - A)^{-1}$$

Exercise 9.8. *Recall that we had* $K_{LQG} = -K(sI - A + BK + FC)^{-1}F$*. Show that this is the same as*

$$K_{LQG} = [I + K(sI - A + FC)^{-1}B]^{-1}K(sI - A + FC)^{-1}F$$